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Axisymmetric solutions of the 3D Navier–Stokes equations for compressible isentropic fluids

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Abstract

We prove the existence of global weak solutions to the Navier–Stokes equations for compressible isentropic fluids for any $\gamma > 1$ when the Cauchy data are axisymmetric, where γ is the specific heat ratio. Moreover, we obtain a new integrability estimate of the density in any neighborhood of the symmetric axis (the singularity axis).

Résumé

On montre l'existence de solutions faibles globales pour le système de Navier–Stokes compressible dans le régime isentropique (pour un $\gamma > 1$) lorsque la donnée initiale est axisymétrique. De plus on obtient une nouvelle estimation sur la densité au voisinage de l'axe de symétrie.

MSC: 76N10; 35M10; 35Q30

Keywords: 3D compressible Navier–Stokes equations; Axisymmetric; Global weak solutions; Weak convergence; Concentration compactness

1. Introduction

This paper is concerned with the global existence of axisymmetric weak solutions to the Cauchy problem for the compressible isentropic Navier–Stokes equations in $\mathbb{R}^3$.

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For axisymmetric flow, there is no flow in the θ-direction and all θ derivatives are identically zero. So we consider only two variables, r the radial direction and z the axial direction. We denote the fluid density respectively the fluid velocity by ρ respectively $\boldsymbol{u} = (u_1, u_2)$, u_1 and u_2 are the radial and axial components of the velocity, respectively. The Navier–Stokes equations for axisymmetric isentropic flow are:

$$\begin{aligned}
&\partial_t \rho + \frac{1}{r}\partial_r(r\rho u_1) + \partial_z(\rho u_2) = 0,\\
&\partial_t(\rho u_1) + \frac{1}{r}\partial_r\big(r\rho u_1^2\big) + \partial_z(\rho u_1 u_2) = -a\partial_r\rho^\gamma + \mu\left[\frac{1}{r}\partial_r(r\partial_r u_1) + \partial_z^2 u_1\right]\\
&\qquad\qquad + \tilde{\mu}\partial_r\left[\frac{1}{r}\partial_r(ru_1) + \partial_z u_2\right] - \mu\frac{u_1}{r^2},\\
&\partial_t(\rho u_2) + \frac{1}{r}\partial_r(r\rho u_1 u_2) + \partial_z\big(\rho u_2^2\big) = -a\partial_z\rho^\gamma + \mu\left[\frac{1}{r}\partial_r(r\partial_r u_2) + \partial_z^2 u_2\right]\\
&\qquad\qquad + \tilde{\mu}\partial_z\left[\frac{1}{r}\partial_r(ru_1) + \partial_z u_2\right],
\end{aligned} \tag{1.1}$$

together with initial and boundary conditions:

$$\rho(0,r,z) = \rho_0, \qquad \rho\boldsymbol{u}(0,r,z) = \boldsymbol{m}_0, \quad (r,z) \in \mathbb{R}^+ \times \mathbb{R}, \tag{1.2}$$

$$u_1(t,0,z) = 0, \qquad \partial_r u_2(t,0,z) = 0, \quad t \geqslant 0,\ z \in \mathbb{R}, \tag{1.3}$$

where $\mu, \tilde{\mu}, a > 0$ and $\gamma > 1$ are constants, $a\rho^\gamma$ is the pressure, $\boldsymbol{m}_0 = (m_0^1, m_0^2)$.

The Navier–Stokes equations for compressible fluids have been studied by many authors. The question concerning the global existence and the time-asymptotic behavior of solutions for large initial data has been largely solved in one dimension (see, e.g., [1,15]). The mathematical theory, however, is far from being complete in more than one dimension. In the case of sufficiently small initial data, there is an extensive literature on the global existence and the asymptotic behavior of solutions which is originated by the papers of Matsumura and Nishida [13,14] (also see [8]). For large initial data with spherical symmetry the global existence was investigated in [7,9]. In the truly multidimensional case, recently, Lions [12] used the weak convergence method and first showed the existence of global weak solutions for isentropic flow under the assumption that $\gamma \geqslant 3/2$ if $n = 2$ and $\gamma \geqslant 9/5$ if $n = 3$, and in [10] the authors proved the global existence of spherically symmetric weak solutions to the Cauchy problem for any $\gamma > 1$. In a recent paper [5], by applying the ideas of the key Lemma 3.2 in [10] and the renormalized solutions of DiPerna and Lions, Feireisl, Novotný and Petzeltová extended Lions' global existence result in $\mathbb{R}^3$ to the case $\gamma > 3/2$. We also mention that the existence of weak time periodic solutions was proved in [4] under a condition on γ similar to that of Lions [12] and the global existence of strong large solutions in [16] under the condition that the viscosity depends on ρ in a very specific way. In [17], non-existence results of global smooth solutions were discussed for initial density with vacuum.

In this paper, we shall combine the ideas in [12,10,5] to prove the global existence of axisymmetric solutions to the 3D compressible isentropic Navier–Stokes equations for any $\gamma > 1$. Comparing with the 2D or the spherically symmetric case, the difficulties here lie in the singularity at $r = 0$, the fact that the singularity set here is the plane $\mathbb{R}_0^+ \times \{0\} \times \mathbb{R}$ in $\mathbb{R}^3$ but not a line as in the spherically symmetric case, and the Neumann boundary condition for u_2 which could induce concentration of singularity involving u_2 at $r = 0$ in passing to the limit $r \downarrow 0$. To exclude the possible concentration of singularity induced by such difficulties, besides modifying the ideas in [12,10,5], we have to adapt Lions' concentration compactness arguments for the stationary isothermal flow (cf. the end of Section 3). Moreover, we obtain a new integrability estimate of the density near $r = 0$ in this paper (i.e., (1.10)).

For the sake of simplicity of the presentation, let us assume that $\tilde{\mu} \equiv 0$. It is easy to see, from the proof throughout this paper, that the case $\tilde{\mu} > 0$ will not arouse any new difficulties.

Now we modify the definition of the so-called finite energy solutions to the system (1.1)–(1.3) in [5] in the following way:

Definition 1.1. We call $(\rho(t, r, z), \boldsymbol{u}(t, r, z))$ $(\boldsymbol{u} = (u_1, u_2))$ a finite energy weak solution of (1.1)–(1.3), if:

(1) $\rho \geqslant 0$ a.e., and for any $T > 0$,

$$
\begin{aligned}
&\rho \in L^\infty\big([0, T], \mathcal{L}^\gamma(\mathbb{R}^+ \times \mathbb{R})\big), \quad \rho|\boldsymbol{u}|^2 \in L^\infty\big([0, T], \mathcal{L}^1(\mathbb{R}^+ \times \mathbb{R})\big),\\
&\nabla \boldsymbol{u}, \quad u_1/r \in L^2\big([0, T], \mathcal{L}^2(\mathbb{R}^+ \times \mathbb{R})\big),\\
&\rho \in C^0\big([0, T], \mathcal{L}^\gamma_{\mathrm{loc}}\big(\mathbb{R}_0^+ \times \mathbb{R}\big) - w\big), \quad \rho\boldsymbol{u} \in C^0\big([0, T], \mathcal{L}^{2\gamma/(\gamma+1)}_{\mathrm{loc}}\big(\mathbb{R}_0^+ \times \mathbb{R}\big) - w\big),\\
&(\rho, \rho\boldsymbol{u})(0, x) = (\rho_0, m_0)(x) \quad \text{weakly in } \mathcal{L}^\gamma_{\mathrm{loc}}\big(\mathbb{R}_0^+ \times \mathbb{R}\big) \times \mathcal{L}^{2\gamma/(\gamma+1)}_{\mathrm{loc}}\big(\mathbb{R}_0^+ \times \mathbb{R}\big).
\end{aligned} \tag{1.4}
$$

(2) For any $b \in C^1(\mathbb{R})$ such that $|b(s)| + |b'(s)s| \leqslant C$ for all $s \in \mathbb{R}$, there holds:

$$
\partial_t b(\rho) + \frac{1}{r}\partial_r\big[r b(\rho) u_1\big] + \partial_z\big[b(\rho)u_2\big] + \big[b'(\rho)\rho - b(\rho)\big]\left(\frac{u_1}{r} + \operatorname{div}\boldsymbol{u}\right) = 0 \tag{1.5}
$$

in $\mathcal{D}'((0, T) \times \mathbb{R}^+ \times \mathbb{R})$, i.e., $(\rho, \boldsymbol{u})$ is a renormalized solution of $(1.1)_1$ (see DiPerna and Lions [2]).

(3) For any $t_2 \geqslant t_1 \geqslant 0$ and any $\psi \in C_0^1(\mathbb{R}^3)$, $\varphi \in C_0^1(\mathbb{R}^3)$ with $\varphi(t, 0, z) = 0$, $\phi \in C_0^1(\mathbb{R}^3)$ with $\phi_r(t, 0, z) = 0$, the following equations hold:

$$
\int\limits_{\mathbb{R}^+ \times \mathbb{R}} \rho u_1 \psi r \,\mathrm{d}r\,\mathrm{d}z\Big|_{t_1}^{t_2} - \int\limits_{t_1}^{t_2} \int\limits_{\mathbb{R}^+ \times \mathbb{R}} (\rho\psi_t + \rho u_1 \psi_r + \rho u_2 \psi_z) r \,\mathrm{d}r\,\mathrm{d}z\,\mathrm{d}t = 0. \tag{1.6}
$$

$$\int\limits_{\mathbb{R}^+\times\mathbb{R}} \rho u_1 \varphi r\,\mathrm{d}r\,\mathrm{d}z\Big|_{t_1}^{t_2} - \int\limits_{t_1}^{t_2}\int\limits_{\mathbb{R}^+\times\mathbb{R}} \big\{\rho u_1\varphi_t + \rho u_1^2\varphi_r + \rho u_1 u_2 \varphi_z\big\} r\,\mathrm{d}r\,\mathrm{d}z\,\mathrm{d}t$$
$$= \int\limits_{t_1}^{t_2}\int\limits_{\mathbb{R}^+\times\mathbb{R}} \left\{a\rho^\gamma\left[\frac{\varphi}{r}+\varphi_r\right] - \mu\partial_r u_1\varphi_r - \mu\frac{u_1}{r^2}\varphi\right\} r\,\mathrm{d}r\,\mathrm{d}z\,\mathrm{d}t; \tag{1.7}$$

$$\int\limits_{\mathbb{R}^+\times\mathbb{R}} \rho u_2 \phi r\,\mathrm{d}r\,\mathrm{d}z\Big|_{t_1}^{t_2} - \int\limits_{t_1}^{t_2}\int\limits_{\mathbb{R}^+\times\mathbb{R}} \big\{\rho u_2\phi_t + \rho u_1 u_2\phi_r + \rho u_2^2 \phi_z\big\} r\,\mathrm{d}r\,\mathrm{d}z\,\mathrm{d}t$$
$$= \int\limits_{t_1}^{t_2}\int\limits_{\mathbb{R}^+\times\mathbb{R}} \big\{a\rho^\gamma\phi_z - \mu\partial_r u_2\phi_r - \mu\partial_z u_2\phi_z\big\} r\,\mathrm{d}r\,\mathrm{d}z\,\mathrm{d}t; \tag{1.8}$$

(4)

$$\int\limits_{\mathbb{R}^+\times\mathbb{R}} \left(\rho\frac{|\boldsymbol{u}|^2}{2} + \frac{a\rho^\gamma}{\gamma-1}\right)(t,r,z) r\,\mathrm{d}r\,\mathrm{d}z + \mu\int\limits_0^t\int\limits_{\mathbb{R}^+\times\mathbb{R}} \left(|\nabla\boldsymbol{u}|^2 + \frac{u_1^2}{r^2}\right) r\,\mathrm{d}r\,\mathrm{d}z\,\mathrm{d}t$$
$$\leqslant \int\limits_{\mathbb{R}^+\times\mathbb{R}} \left(\frac{|\boldsymbol{m}_0|^2}{2\rho_0} + \frac{a\rho_0^\gamma}{\gamma-1}\right) r\,\mathrm{d}r\,\mathrm{d}z \quad \forall t\geqslant 0. \tag{1.9}$$

Then, the main result of this paper reads:

Theorem 1.1. *Let* $\gamma > 1$, $0 \leqslant \rho_0 \in \mathcal{L}^\gamma(\mathbb{R}^+\times\mathbb{R}) \cap \mathcal{L}^1(\mathbb{R}^+\times\mathbb{R})$ *and* $\boldsymbol{m}_0/\sqrt{\rho_0} \in \mathcal{L}^2(\mathbb{R}^+\times\mathbb{R})$. *Then there exists a global weak solution of* (1.1)–(1.3)*, such that for any* $T, L > 0$ *and* $\alpha \in (0,1)$*,*

$$\int\limits_0^T\int\limits_0^1\int\limits_{-L}^L \big(\rho^\gamma + \rho u_1^2\big) r^\alpha\,\mathrm{d}r\,\mathrm{d}z\,\mathrm{d}t \leqslant C. \tag{1.10}$$

Remark 1.1. (i) If we define $\rho(t,\boldsymbol{x}) := \rho(t,r,z)$, $\boldsymbol{U}(t,\boldsymbol{x}) := ((\boldsymbol{x}'/r)u_1(t,r,z), u_2(t,r,z))$, where $\boldsymbol{x} = (x,y,z) \in \mathbb{R}^3$, $\boldsymbol{x}' = (x,y) \in \mathbb{R}^2$ and $r = |\boldsymbol{x}'|$. Then it is easy to see that $(\rho(t,\boldsymbol{x}), \boldsymbol{U}(t,\boldsymbol{x}))$ is a weak solution of the Cauchy problem for the compressible isentropic Navier–Stokes equations in $\mathbb{R}^3$ (cf. the proof of Theorem 5.7 in [7]).

We shall give the proof of Theorem 1.1 in Section 3. In Section 2 we derive a priori estimates for the approximate weak solutions and in Section 4 we prove the global existence of the approximate weak solutions.

Notation (*used throughout this paper*). Let Ω be a domain in $\mathbb{R}^2$. Let m be an integer and let $1 \leqslant p \leqslant \infty$. By $W^{m,p}(\Omega)$ $(W_0^{m,p}(\Omega))$ we denote the usual Sobolev space defined over Ω. $W^{m,2}(\Omega) \equiv H^m(\Omega)$ ($W_0^{m,2}(\Omega) \equiv H_0^m(\Omega)$), $W^{0,p}(\Omega) \equiv L^p(\Omega)$ with norm $\|\cdot\|_{L^p(\Omega)}$. We define:

$$\mathcal{L}^p(\Omega) := \left\{ f \in L^1_{\text{loc}}(\Omega)\colon \int_\Omega \left|f(r,z)\right|^p r \, \mathrm{d}r \, \mathrm{d}z < \infty \right\}$$

with norm $\|\cdot\|_{\mathcal{L}^p(\Omega)} := (\int_\Omega |\cdot|^p r \, \mathrm{d}r \, \mathrm{d}z)^{1/p}$. $\mathcal{L}^p_{\text{loc}}(\Omega)$ and $\mathcal{H}^1_{\text{loc}}(\Omega)$ are defined similarly to $L^p_{\text{loc}}(\Omega)$ and $H^1_{\text{loc}}(\Omega)$, respectively. For simplicity we also use the following abbreviations:

$$\|\cdot\|_{\mathcal{L}^p} \equiv \|\cdot\|_{\mathcal{L}^p(\mathbb{R}^+\times\mathbb{R})}, \qquad \|\cdot\|_{L^p} \equiv \|\cdot\|_{L^p(\mathbb{R}^+\times\mathbb{R})}, \qquad \mathbb{R}^+ := (0,\infty),$$
$$\mathbb{R}^+_0 := [0,\infty), \qquad \nabla := (\partial_r, \partial_z), \qquad \operatorname{div} := \nabla\cdot, \qquad \Delta := \partial_r^2 + \partial_z^2.$$

$L^p(I,B)$ respectively $\|\cdot\|_{L^p(I,B)}$ denotes the space of all strongly measurable, pth-power integrable (essentially bounded if $p = \infty$) functions from I to B respectively its norm, $I \subset \mathbb{R}$ an interval, B a Banach space. $C^0(I, B-w)$ is the space of all functions which are in $L^\infty(I,B)$ and continuous in t with values in B endowed with the weak topology.

The same letter C (sometimes used as $C(X)$ to emphasize the dependence of C on X) will denote various positive constants which do not depend on ε and δ.

2. A priori estimates for the approximate solutions

We shall construct approximate solutions of (1.1)–(1.3) by adding an artificial pressure term $\varepsilon^\lambda \rho^\beta$ ($\beta > \max\{4,\gamma\}$, $\lambda > 3\beta/\gamma - 3$) and cutting off the singularity induced by the origin in (1.1).

We start with construction of an approximation to the initial data $\rho_0, \boldsymbol{m}_0$. Let $\chi_1^\varepsilon, \chi_2^\varepsilon \in C^\infty(\mathbb{R})$ satisfy $\chi_1^\varepsilon(x) = 1$ for $x \leqslant \varepsilon^{-2}$, $\chi_2^\varepsilon(r) = 1$ for $r \geqslant 3\varepsilon$, and $\chi_1^\varepsilon(x) = 0$ for $x \geqslant 2\varepsilon^{-2}$, $\chi_2^\varepsilon(r) = 0$ for $r \leqslant 2\varepsilon$. We define:

$$\rho_0^\varepsilon(r,z) := r^{-1/\gamma}\left[\left(r^{1/\gamma}\rho_0\right) * j_{\varepsilon/2}\right](r,z)\chi_1^\varepsilon\left(r^2+z^2\right),$$
$$\boldsymbol{m}_0^\varepsilon(r,z) := \chi_2^\varepsilon(r)\cdot \begin{cases} \left[\left(\boldsymbol{m}_0/\sqrt{\rho_0}\right) * j_\varepsilon\right](r,z)\sqrt{\rho_0^\varepsilon(r,z)}, & \rho_0(r,z) > 0,\\ 0, & \rho_0(r,z) = 0,\end{cases}$$

where $j_\varepsilon = \frac{1}{\varepsilon^3} j(r/\varepsilon, \theta/\varepsilon)$ with $\int_0^\infty \int_{\mathbb{R}} j(r,\theta) r \, \mathrm{d}r \, \mathrm{d}\theta = 1$. Thus, the approximate solutions are obtained by solving the following initial boundary value problem in the domain $(\varepsilon,\infty)\times\mathbb{R}$:

$$\partial_t \rho^\varepsilon + \frac{1}{r}\partial_r\left(r\rho^\varepsilon u_1^\varepsilon\right) + \partial_z\left(\rho^\varepsilon u_2^\varepsilon\right) = 0, \tag{2.1}$$

$$\begin{aligned}\partial_t\left(\rho^\varepsilon u_1^\varepsilon\right) + \frac{1}{r}\partial_r\left[r\rho^\varepsilon\left(u_1^\varepsilon\right)^2\right] + \partial_z\left[\rho^\varepsilon u_1^\varepsilon u_2^\varepsilon\right] &= -a\partial_r(\rho^\varepsilon)^\gamma - \varepsilon^\lambda\partial_r(\rho^\varepsilon)^\beta\\ &\quad + \mu\left[\frac{1}{r}\partial_r\left(r\partial_r u_1^\varepsilon\right) + \partial_z^2 u_1^\varepsilon\right] - \mu\frac{u_1^\varepsilon}{r^2},\end{aligned} \tag{2.2}$$

$$\partial_t\left(\rho^\varepsilon u_2^\varepsilon\right)+\frac{1}{r}\partial_r\left(r\rho^\varepsilon u_1^\varepsilon u_2^\varepsilon\right)+\partial_z\left[\rho^\varepsilon\left(u_2^\varepsilon\right)^2\right]=-a\partial_z(\rho^\varepsilon)^\gamma-\varepsilon^\lambda\partial_z(\rho^\varepsilon)^\beta + \mu\left[\frac{1}{r}\partial_r\left(r\partial_r u_2^\varepsilon\right)+\partial_z^2 u_2^\varepsilon\right] \tag{2.3}$$

together with initial and boundary conditions:

$$\rho^\varepsilon(0,r,z)=\rho_0^\varepsilon,\qquad \rho^\varepsilon\boldsymbol{u}^\varepsilon(0,r,z)=\boldsymbol{m}_0^\varepsilon,\quad (r,z)\in(\varepsilon,\infty)\times\mathbb{R}, \tag{2.4}$$

$$u_1^\varepsilon(t,\varepsilon,z)=0,\qquad \partial_r u_2^\varepsilon(t,\varepsilon,z)=0,\quad t\geqslant 0,\ z\in\mathbb{R}, \tag{2.5}$$

where $\boldsymbol{u}^\varepsilon=(u_1^\varepsilon,u_2^\varepsilon)$, $\beta>\max\{4,\gamma\}$ and $\lambda>3\beta/\gamma-3$ are constants.

In view of the definition of ρ_0^ε, $\boldsymbol{m}_0^\varepsilon$, it is easy to see that $\rho_0^\varepsilon\in\mathcal{L}^\beta(\mathbb{R}^+\times\mathbb{R})$, $\rho_0^\varepsilon\geqslant 0$ a.e., and

$$\left\|\rho_0^\varepsilon-\rho_0\right\|_{\mathcal{L}^\gamma(\mathbb{R}^+\times\mathbb{R})}\to 0,\qquad \left\|\frac{\boldsymbol{m}_0^\varepsilon}{\sqrt{\rho_0^\varepsilon}}-\frac{\boldsymbol{m}_0}{\sqrt{\rho_0}}\right\|_{\mathcal{L}^2(\mathbb{R}^+\times\mathbb{R})}\to 0\quad\text{as }\varepsilon\to 0,$$
$$\int\limits_{(\varepsilon,\infty)\times\mathbb{R}}\rho_0^\varepsilon r\,\mathrm{d}r\,\mathrm{d}z\leqslant C\int\limits_{\mathbb{R}^+\times\mathbb{R}}\rho_0 r\,\mathrm{d}r\,\mathrm{d}z. \tag{2.6}$$

Hence, by virtue of Theorem 4.3 in Section 4, the problem (2.1)–(2.5) has a global weak solution $(\rho^\varepsilon,u_1^\varepsilon,u_2^\varepsilon)$ on $\mathbb{R}_0^+\times[\varepsilon,\infty)\times\mathbb{R}$ with $\rho^\varepsilon\geqslant 0$ a.e., such that

$$\int_\varepsilon^\infty\int_{\mathbb{R}}\left[\rho^\varepsilon\frac{|\boldsymbol{u}^\varepsilon|^2}{2}+\frac{a(\rho^\varepsilon)^\gamma}{\gamma-1}+\frac{\varepsilon^\lambda(\rho^\varepsilon)^\beta}{\beta-1}\right]r\,\mathrm{d}r\,\mathrm{d}z+\mu\int_0^t\int_\varepsilon^\infty\int_{\mathbb{R}}\left[|\nabla\boldsymbol{u}^\varepsilon|^2+\frac{(u_1^\varepsilon)^2}{r^2}\right]r\,\mathrm{d}r\,\mathrm{d}z\,\mathrm{d}\tau$$
$$\leqslant\int_\varepsilon^\infty\int_{\mathbb{R}}\left[\frac{|\boldsymbol{m}_0^\varepsilon|^2}{2\rho_0^\varepsilon}+\frac{a(\rho_0^\varepsilon)^\gamma}{\gamma-1}+\frac{\varepsilon^\lambda(\rho_0^\varepsilon)^\beta}{\beta-1}\right]r\,\mathrm{d}r\,\mathrm{d}z\quad\forall t\geqslant 0, \tag{2.7}$$

$$\int_\varepsilon^\infty\int_{\mathbb{R}}\rho^\varepsilon(t,r,z)r\,\mathrm{d}r\,\mathrm{d}z\leqslant\int\limits_{\mathbb{R}^+\times\mathbb{R}}\rho_0 r\,\mathrm{d}r\,\mathrm{d}z\quad\forall t\geqslant 0. \tag{2.8}$$

Notice that for $\beta>2$, by the proof in [12], ρ^ε is in fact a renormalized solution of (2.1), that is, for any $b\in C^1(\mathbb{R})$, $|b(s)|\leqslant C$ and $|b'(s)s|\leqslant C$, we have:

$$\partial_t b(\rho^\varepsilon)+\frac{1}{r}\partial_r\left[rb(\rho^\varepsilon)u_1^\varepsilon\right]+\partial_z\left[b(\rho^\varepsilon)u_2^\varepsilon\right]+\left(b'(\rho^\varepsilon)\rho^\varepsilon-b(\rho^\varepsilon)\right)\left(\frac{u_1^\varepsilon}{r}+\operatorname{div}\boldsymbol{u}^\varepsilon\right)=0. \tag{2.9}$$

Moreover, by Hölder's inequality and the fact that $\lambda>3\beta/\gamma-3$, we have:

$$\begin{aligned}\varepsilon^\lambda \|\rho_0^\varepsilon\|^\beta_{\mathcal{L}^\beta((\varepsilon,\infty)\times\mathbb{R})} &\leqslant C\varepsilon^{1-\beta/\gamma+\lambda}\big\|\big(r\rho_0^\gamma\big)*j_{\varepsilon/2}\big\|^{\beta/\gamma}_{L^{\beta/\gamma}}\\ &\leqslant C\varepsilon^{3-3\beta/\gamma+\lambda}\|\rho_0\|^{\beta-\gamma}_{\mathcal{L}^\gamma}\big\|\big(r\rho_0^\gamma\big)*j_{\varepsilon/2}\big\|_{L^1}\\ &\leqslant C\varepsilon^{3-3\beta/\gamma+\lambda}\|\rho_0\|^\beta_{\mathcal{L}^\gamma}\to 0 \quad \text{as } \varepsilon\to 0, \end{aligned}\tag{2.10}$$

which together with (2.6) and (2.7) gives:

$$\text{L.H.S. of (2.7)} \leqslant C\left\{1+\int_{\mathbb{R}^+\times\mathbb{R}}\left(\frac{|\boldsymbol{m}_0|^2}{2\rho_0}+\frac{a\rho_0^\gamma}{\gamma-1}\right)r\,\mathrm{d}r\,\mathrm{d}z\right\}\quad \forall t\geqslant 0.\tag{2.11}$$

Using (2.7) and approximating $b(\rho)$ by ρ^θ with $0<\theta\leqslant\gamma-1$ in (2.9) (i.e., taking some $b_R\in C^1(\mathbb{R})$ with $|b_R(s)|+|b_R'(s)s|\leqslant C$ and $b_R(s)\to s^\theta$ as $R\to\infty$), we find that (also cf. [12, pp. 15,19]),

$$\partial_t(\rho^\varepsilon)^\theta+\frac{1}{r}\partial_r\big[r(\rho^\varepsilon)^\theta u_1^\varepsilon\big]+\partial_z\big[(\rho^\varepsilon)^\theta u_2^\varepsilon\big]+(\theta-1)(\rho^\varepsilon)^\theta\left(\frac{u_1^\varepsilon}{r}+\operatorname{div}\boldsymbol{u}^\varepsilon\right)=0.\tag{2.12}$$

Then, multiplying (2.2)–(2.3) by $\varphi\in C_0^\infty((\varepsilon,\infty))$ and utilizing (2.12), we obtain by similar calculations in [12, Chapter 5] that

$$\begin{aligned}&\varphi(r)(\rho^\varepsilon)^\theta\big\{a(\rho^\varepsilon)^\gamma+\varepsilon^\lambda(\rho^\varepsilon)^\beta-\mu\operatorname{div}\boldsymbol{u}^\varepsilon\big\}\\ &\quad=\partial_t\big\{(\rho^\varepsilon)^\theta(-\Delta)^{-1}\operatorname{div}\big(\varphi(r)\rho^\varepsilon\boldsymbol{u}^\varepsilon\big)\big\}+(\rho^\varepsilon)^\theta R_{ij}\big(\varphi(r)\rho^\varepsilon u_i^\varepsilon u_j^\varepsilon\big)\\ &\qquad-(\rho^\varepsilon)^\theta u_i^\varepsilon R_{ij}\big(\varphi(r)\rho^\varepsilon u_j^\varepsilon\big)+\operatorname{div}\big\{(\rho^\varepsilon)^\theta\boldsymbol{u}^\varepsilon(-\Delta)^{-1}\operatorname{div}\big[\varphi(r)\rho^\varepsilon\boldsymbol{u}^\varepsilon\big]\big\}\\ &\qquad+(-\Delta)^{-1}\operatorname{div}\big(\varphi(r)\rho^\varepsilon\boldsymbol{u}^\varepsilon\big)\left\{\theta\frac{(\rho^\varepsilon)^\theta u_1^\varepsilon}{r}+(\theta-1)(\rho^\varepsilon)^\theta\operatorname{div}\boldsymbol{u}^\varepsilon\right\}\\ &\qquad-(\rho^\varepsilon)^\theta(-\Delta)^{-1}\partial_r\left\{r\rho^\varepsilon\big(u_1^\varepsilon\big)^2\partial_r\left[\frac{\varphi}{r}\right]\right\}-(\rho^\varepsilon)^\theta(-\Delta)^{-1}\partial_z\left\{r\rho^\varepsilon u_1^\varepsilon u_2^\varepsilon\partial_r\left[\frac{\varphi}{r}\right]\right\}\\ &\qquad-a(\rho^\varepsilon)^\theta(-\Delta)^{-1}\partial_r\big[(\rho^\varepsilon)^\gamma\partial_r\varphi\big]-\varepsilon^\lambda(\rho^\varepsilon)^\theta(-\Delta)^{-1}\partial_r\big[(\rho^\varepsilon)^\beta\partial_r\varphi\big]\\ &\qquad+\mu(\rho^\varepsilon)^\theta(-\Delta)^{-1}\partial_r\left\{r\partial_r u_1^\varepsilon\partial_r\left[\frac{\varphi}{r}\right]\right\}-\mu(\rho^\varepsilon)^\theta(-\Delta)^{-1}\big\{\partial_z^2u_1^\varepsilon\partial_r\varphi\big\}\\ &\qquad+\mu(\rho^\varepsilon)^\theta(-\Delta)^{-1}\partial_{rz}^2\big(u_2^\varepsilon\partial_r\varphi\big)+\mu(\rho^\varepsilon)^\theta(-\Delta)^{-1}\partial_z\left\{r\partial_r u_2^\varepsilon\partial_r\left[\frac{\varphi}{r}\right]\right\}\\ &\qquad+\mu(\rho^\varepsilon)^\theta(-\Delta)^{-1}\partial_r\left[\frac{u_1^\varepsilon}{r^2}\varphi\right]\quad(0<\theta\leqslant\gamma-1),\end{aligned}\tag{2.13}$$

where $R_{ij}=(-\Delta)^{-1}\partial_i\partial_j$ with $(\partial_1,\partial_2)=(\partial_r,\partial_z)$ is the Riesz transform. Thus, multiplying (2.13) with $\theta=\gamma-1$ by r and integrating, following the same process as in the proof of Theorem 7.1 in [12, Chapter 7], we can obtain that

$$\int_0^T \int_{\mathbb{R}^+\times\mathbb{R}} \varphi(r)\big\{a(\rho^\varepsilon)^{\gamma+\theta} + \varepsilon^\lambda(\rho^\varepsilon)^{\beta+\theta}\big\} r \,\mathrm{d}r\,\mathrm{d}z\,\mathrm{d}t \leqslant C, \quad \theta = \gamma - 1, \tag{2.14}$$

where C is a positive constant depending only on θ, ρ_0 and $\boldsymbol{m}_0$.

Remark 2.1. The condition $\beta > \max\{4, \gamma\}$ is needed in order to apply Theorem 4.3 in Section 4. In fact, $\beta > \max\{2, \gamma\}$ is sufficient to get (2.14) and the precompactness shown in the next section.

Next we exploit the pressure term in (2.2) to derive a (better) integrability estimate of ρ^ε near $r = 0$, which will be needed at the end of Section 3.

For $L > \varepsilon$, $h > 0$ let $\phi \in C_0^1(\mathbb{R}^3)$ and $\chi^h \in C^\infty((\varepsilon, \infty))$ be non-negative such that

$$\phi(t, r, z) = 1 \quad \text{for } (t, r, z) \in [0, T] \times [0, 1] \times [-L, L],$$
$$\chi^h(r) = \begin{cases} 0, & \varepsilon \leqslant r \leqslant h + \varepsilon, \\ 1, & \varepsilon + 2h \leqslant r, \end{cases} \qquad 0 \leqslant \partial_r \chi^h(r) \leqslant Ch^{-1}.$$

Taking $\varphi(t, r, z) = r(r - \varepsilon)^{2/3}\phi(t, r, z)\chi^h(r)$ as a test function for (2.2), we multiply (2.2) by φ and integrate over $(0, T) \times G_\varepsilon$, where $G_\varepsilon := (\varepsilon, \infty) \times \mathbb{R}$. Integrating then by parts and making use of Cauchy–Schwartz's inequality, taking into account that

$$\varphi_r \geqslant \frac{2}{3} r^{2/3}\phi\chi^h + r(r - \varepsilon)^{2/3}\chi^h\phi_r \quad \text{and}$$
$$\partial_r(\varphi/r) \geqslant (r - \varepsilon)^{2/3}\chi^h\phi_r + \frac{2}{3}(r - \varepsilon)^{-1/3}\chi^h\phi,$$

we find that

$$\begin{aligned}
&\frac{2}{3}\int_0^T \int_{G_\varepsilon} \big\{a(\rho^\varepsilon)^\gamma + \varepsilon^\lambda(\rho^\varepsilon)^\beta + \rho^\varepsilon\big(u_1^\varepsilon\big)^2\big\} r^{2/3}\phi\chi^h \,\mathrm{d}r\,\mathrm{d}z\,\mathrm{d}t \\
&\leqslant 2 \sup_{0\leqslant t\leqslant T} \int_{G_\varepsilon} \rho^\varepsilon\big|u_1^\varepsilon\big|\varphi(t, r, z)\,\mathrm{d}r\,\mathrm{d}z \\
&\quad + \int_0^T \int_\Omega \left\{\rho^\varepsilon\big|u_1^\varepsilon\varphi_t\big| + \rho^\varepsilon\big|u_1^\varepsilon u_2^\varepsilon\varphi_z\big| + \mu\big|\partial_z u_1^\varepsilon\varphi_z\big| + \mu\frac{|u_1^\varepsilon|\varphi}{r^2}\right\}\mathrm{d}r\,\mathrm{d}z\,\mathrm{d}t \\
&\quad + \int_0^T \int_{G_\varepsilon} \Big\{\big[\rho^\varepsilon\big(u_1^\varepsilon\big)^2 + a(\rho^\varepsilon)^\gamma + \varepsilon^\lambda(\rho^\varepsilon)^\beta\big]|\phi_r|\chi^h \\
&\qquad\qquad + \mu\big|\partial_r u_1^\varepsilon\big|\left[\big|\partial_r\big(\phi\chi^h\big)\big| + \frac{2\phi\chi^h}{3(r - \varepsilon)}\right]\Big\}(r - \varepsilon)^{2/3} r \,\mathrm{d}r\,\mathrm{d}z\,\mathrm{d}t,
\end{aligned}$$

whence,

$$
\begin{aligned}
&\int_0^T \int_{G_\varepsilon} \left\{a(\rho^\varepsilon)^\gamma + \varepsilon^\lambda(\rho^\varepsilon)^\beta + \rho^\varepsilon\left(u_1^\varepsilon\right)^2\right\} r^{2/3}\phi\chi^h \,\mathrm{d}r\,\mathrm{d}z\,\mathrm{d}t \\
&\quad \leqslant C \sup_{0\leqslant t\leqslant T} \int_{G_\varepsilon} \left\{\rho^\varepsilon + \rho^\varepsilon|\boldsymbol{u}^\varepsilon|^2 + (\rho^\varepsilon)^\gamma + \varepsilon^\lambda(\rho^\varepsilon)^\beta\right\} r\,\mathrm{d}r\,\mathrm{d}z \\
&\qquad + C\int_0^T\int_{G_\varepsilon} \left\{|\nabla \boldsymbol{u}^\varepsilon|^2 + \frac{(u_1^\varepsilon)^2}{r^2} + \left[\left|\partial_r\left(\phi\chi^h\right)\right| + \frac{\phi\chi^h}{r-\varepsilon}\right]^2 (r-\varepsilon)^{4/3}\right\} r\,\mathrm{d}r\,\mathrm{d}z\,\mathrm{d}t \\
&\quad \leqslant C + C\int_0^T\int_{\mathbb{R}} \left\{\int_{h+\varepsilon}^{2h+\varepsilon} \frac{(r-\varepsilon)^{4/3}}{h^2} + \int_\varepsilon^\infty \frac{1}{(r-\varepsilon)^{2/3}}\right\}\phi r\,\mathrm{d}r\,\mathrm{d}z\,\mathrm{d}t \\
&\quad \leqslant C
\end{aligned}
\tag{2.15}
$$

with C being independent of ε and h. Hence, letting $h \to 0$ in (2.15), we obtain that for any $L, T > 0$,

$$
\int_0^T \int_\varepsilon^1 \int_{-L}^L \left\{(\rho^\varepsilon)^\gamma + \varepsilon^\lambda(\rho^\varepsilon)^\beta + \rho^\varepsilon\left(u_1^\varepsilon\right)^2\right\} r^{2/3}\,\mathrm{d}r\,\mathrm{d}z\,\mathrm{d}t \leqslant C, \tag{2.16}
$$

where the constant C does not depend on ε.

3. Proof of the precompactness

In this section we extract a limiting solution $(\rho, \boldsymbol{u})$ from the approximate weak solution sequence $(\rho^\varepsilon, \boldsymbol{u}^\varepsilon)$ ($\boldsymbol{u}^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon)$) of (2.1)–(2.5), and thus obtain a global weak solution of (1.1)–(1.3).

First we extend $\rho^\varepsilon(t, r, z)$ as well as $u_1^\varepsilon(t, r, z)$ to be zero and $u_2^\varepsilon(t, r, z)$ to be $u_2^\varepsilon(t, \varepsilon, z)$ for $(t, r, z) \in \mathbb{R}_0^+ \times [0, \varepsilon) \times \mathbb{R}$. For simplicity, we still denote by $(\rho^\varepsilon, \boldsymbol{u}^\varepsilon)$ this extension. Throughout this section we denote $\Omega := \mathbb{R}^+ \times \mathbb{R}$.

It follows from (2.11) and (5.91) in Lions' book [12, p. 43] that $\|\boldsymbol{u}^\varepsilon\|_{L^2((0,T),\mathcal{H}^1_{\mathrm{loc}}(\Omega))}$ is uniformly bounded with respect to ε, and hence, we can extract a subsequence of $(\rho^\varepsilon, \boldsymbol{u}^\varepsilon)$, still denoted by $(\rho^\varepsilon, \boldsymbol{u}^\varepsilon)$ with

$$
\rho^\varepsilon \in L^\infty\left([0,T], \mathcal{L}^\gamma(\Omega) \cap \mathcal{L}^1(\Omega)\right), \quad \nabla\boldsymbol{u}^\varepsilon \in L^2\left([0,T], \mathcal{L}^2(\Omega)\right),
$$

such that

$$
\begin{aligned}
&\rho^\varepsilon \rightharpoonup \rho \quad \text{weak-}* \text{ in } L^\infty\left([0,T], \mathcal{L}^\gamma(\Omega)\right), \\
&\boldsymbol{u}^\varepsilon \rightharpoonup \boldsymbol{u} \quad \text{weakly in } L^2\left([0,T], \mathcal{H}^1_{\mathrm{loc}}(\Omega)\right).
\end{aligned}
\tag{3.1}
$$

Using Eq. (2.1) and (2.11), we see that $\partial_t \rho^\varepsilon \in L^2((0,T), W^{-1,p}_{\mathrm{loc}}(\Omega))$ for any $1 < p < \gamma$. So, Appendix C of [11] gives:

$$\rho^\varepsilon \to \rho \quad \text{in } C^0\big([0,T], L^p_{\mathrm{loc}}(\Omega) - w\big) \text{ for any } 1 < p < \gamma. \tag{3.2}$$

On the other hand, since $\gamma > 1$, we can take $1 < p < \gamma$ so that $L^p_{\mathrm{loc}}(\Omega) \hookrightarrow \lambda H^{-1}_{\mathrm{loc}}(\Omega)$, thus for any $T < \infty$, $\rho^\varepsilon \to \rho$ in $C^0([0,T], H^{-1}_{\mathrm{loc}}(\Omega))$ as $\varepsilon \to 0$ by a contradiction argument. This together with (3.2) implies that

$$\rho^\varepsilon \boldsymbol{u}^\varepsilon \rightharpoonup \rho \boldsymbol{u} \quad \text{in } \mathcal{D}'\big((0,T) \times \Omega\big). \tag{3.3}$$

Moreover, by (2.11), (2.2) and (2.3),

$$\rho^\varepsilon \boldsymbol{u}^\varepsilon \in L^\infty\big([0,T], \mathcal{L}^{2\gamma/(\gamma+1)}(\Omega)\big) \cap L^2\big([0,T], L^p_{\mathrm{loc}}(\Omega)\big) \quad \text{for any } p < \gamma$$

and

$$\partial_t(\rho^\varepsilon \boldsymbol{u}^\varepsilon) \in L^1\big([0,T], W^{-1,\tilde{p}}_{\mathrm{loc}}(\Omega)\big) \quad \text{for some } 1 < \tilde{p} < \min\{\gamma, 2\}.$$

Hence, Appendix C of [11] and (3.3) in fact imply:

$$\begin{aligned} \rho^\varepsilon \boldsymbol{u}^\varepsilon \rightharpoonup \rho \boldsymbol{u} \quad & \text{weak-}{*} \text{ in } L^\infty\big([0,T], \mathcal{L}^{2\gamma/(\gamma+1)}(\Omega)\big) \text{ and} \\ & \text{weakly in } L^2\big([0,T], L^p_{\mathrm{loc}}(\Omega)\big) \text{ for any } p < \gamma, \\ \rho^\varepsilon \boldsymbol{u}^\varepsilon \to \rho \boldsymbol{u} \quad & \text{in } C^0\big([0,T], L^p_{\mathrm{loc}}(\Omega) - w\big) \text{ for any } p \leqslant \frac{2\gamma}{\gamma+1}. \end{aligned} \tag{3.4}$$

Furthermore, from (3.1), (3.2), (3.4) and a density argument, we can get $(1.4)_3$, $(1.4)_4$.

Similarly to (3.3) and (3.4), we have:

$$\rho^\varepsilon \boldsymbol{u}^\varepsilon \otimes \boldsymbol{u}^\varepsilon \rightharpoonup \rho \boldsymbol{u} \otimes \boldsymbol{u} \quad \text{weakly in } L^2\big([0,T], L^p_{\mathrm{loc}}(\Omega)\big) \text{ for any } 1 < p < \frac{2\gamma}{\gamma+1}. \tag{3.5}$$

Moreover, from (3.1), (3.5), and (2.6), (2.7), (2.10) and the lower semicontinuity of weak convergence, the estimate (1.9) follows.

By (2.14) and Hölder's inequality, we conclude that for any $K \Subset \mathbb{R}^+ \times \mathbb{R}$,

$$\begin{aligned} &\varepsilon^\lambda \int_0^T \int_K (\rho^\varepsilon)^\beta \,\mathrm{d}r\,\mathrm{d}z\,\mathrm{d}t \\ &\quad \leqslant C(K)\varepsilon^{\lambda\theta/(\beta+\theta)} \left\{ \varepsilon^\lambda \int_0^T \int_K (\rho^\varepsilon)^{\beta+\theta} r \,\mathrm{d}r\,\mathrm{d}z\,\mathrm{d}t \right\}^{\beta/(\beta+\theta)} \to 0 \quad \text{as } \varepsilon \to 0. \end{aligned} \tag{3.6}$$

Summing up (3.1)–(3.6) and taking $\varepsilon \to 0$ in (2.1)–(2.3), one gets:

$$
\begin{aligned}
&\partial_t\rho+\frac{1}{r}\partial_r(r\rho u_1)+\partial_z(\rho u_2)=0,\\
&\partial_t(\rho u_1)+\frac{1}{r}\partial_r\bigl(r\rho u_1^2\bigr)+\partial_z(\rho u_1u_2)=-a\partial_r\overline{\rho^\gamma}+\mu\left[\frac{1}{r}\partial_r(r\partial_r u_1)+\partial_z^2u_1\right]-\mu\frac{u_1}{r^2},\\
&\partial_t(\rho u_2)+\frac{1}{r}\partial_r(r\rho u_1u_2)+\partial_z\bigl(\rho u_2^2\bigr)=-a\partial_z\overline{\rho^\gamma}+\mu\left[\frac{1}{r}\partial_r(ru_2)+\partial_z^2u_2\right]
\end{aligned}\tag{3.7}
$$

in the sense of $\mathcal{D}'(\mathbb{R}^+\times\mathbb{R}^+\times\mathbb{R})$. Here for the sake of convenience we denote by $\overline{f(\rho)}$ the weak limit of $f(\rho_\varepsilon)$ (in the sense of distributions) as $\varepsilon\to 0$.

Moreover, by arguments similar to those in the proof of Lemma 4.4 in [5], we find that $(\rho,\boldsymbol{u})$ solves $(1.1)_1$ in the sense of renormalized solutions, i.e., Eq. (1.5) *holds*.

Thus, to show that $(\rho,\boldsymbol{u})$ is indeed a finite energy weak solution of (1.1)–(1.3), we need first to prove that $\overline{\rho^\gamma}=\rho^\gamma$. To this end we first apply the same argument as the one in the derivation of (3.3)–(3.4), taking into account that the Riesz operator R_{ij} is bounded from L^p to L^p for any $1<p<\infty$, to deduce that

$$
\begin{aligned}
&(\rho^\varepsilon)^\theta(-\Delta)^{-1}\partial_r\left[r\partial_r u_1^\varepsilon\partial_r\left(\frac{\varphi}{r}\right)\right]\rightharpoonup\overline{\rho^\theta}(-\Delta)^{-1}\partial_r\left[r\partial_r u_1\partial_r\left(\frac{\varphi}{r}\right)\right],\\
&(\rho^\varepsilon)^\theta(-\Delta)^{-1}\bigl\{\partial_z^2u_1^\varepsilon\partial_r\varphi\bigr\}\rightharpoonup\overline{\rho^\theta}(-\Delta)^{-1}\bigl\{\partial_z^2u_1\partial_r\varphi\bigr\},\\
&(\rho^\varepsilon)^\theta(-\Delta)^{-1}\partial_{rz}^2\bigl[u_2^\varepsilon\partial_r\varphi\bigr]\rightharpoonup\overline{\rho^\theta}(-\Delta)^{-1}\partial_{rz}^2[u_2\partial_r\varphi],\\
&(\rho^\varepsilon)^\theta(-\Delta)^{-1}\partial_z\left[r\partial_r u_2^\varepsilon\partial_r\left(\frac{\varphi}{r}\right)\right]\rightharpoonup\overline{\rho^\theta}(-\Delta)^{-1}\partial_z\left[r\partial_r u_2\partial_r\left(\frac{\varphi}{r}\right)\right],\\
&(\rho^\varepsilon)^\theta(-\Delta)^{-1}\partial_r\left[\frac{u_1^\varepsilon}{r^2}\varphi\right]\rightharpoonup\overline{\rho^\theta}(-\Delta)^{-1}\partial_r\left[\frac{u_1}{r^2}\varphi\right],\\
&(\rho^\varepsilon)^\theta\boldsymbol{u}^\varepsilon\rightharpoonup\overline{\rho^\theta}\boldsymbol{u}\qquad\text{in } L^2\bigl([0,T],L^p_{\mathrm{loc}}(\Omega)\bigr)\ \forall p<\frac{\gamma}{\theta}.
\end{aligned}\tag{3.8}
$$

From Eqs. (2.2) and (2.3) we get:

$$
\partial_t(-\Delta)^{-1}\operatorname{div}\bigl[\varphi(r)\rho^\varepsilon\boldsymbol{u}^\varepsilon\bigr]\in L^\infty\bigl([0,T],L^1_{\mathrm{loc}}(\Omega)\bigr)+L^2\bigl([0,T],L^2_{\mathrm{loc}}(\Omega)\bigr).
$$

Therefore, by the classical Lions–Aubin Lemma, we obtain:

$$
(-\Delta)^{-1}\operatorname{div}\bigl[\varphi(r)\rho^\varepsilon\boldsymbol{u}^\varepsilon\bigr]\to(-\Delta)^{-1}\operatorname{div}\bigl[\varphi(r)\rho\boldsymbol{u}\bigr]\quad\text{in } L^q\bigl([0,T],L^p_{\mathrm{loc}}(\Omega)\bigr)\tag{3.9}
$$

for any $1<q<\infty$, $p\leqslant 2\gamma$. Combining (3.8) with (3.9), we conclude:

$$
\begin{gathered}
(\rho^\varepsilon)^\theta\boldsymbol{u}^\varepsilon(-\Delta)^{-1}\operatorname{div}\bigl[\varphi(r)\rho^\varepsilon\boldsymbol{u}^\varepsilon\bigr]\rightharpoonup\overline{\rho^\theta}\boldsymbol{u}(-\Delta)^{-1}\operatorname{div}\bigl[\varphi(r)\rho\boldsymbol{u}\bigr],\\
\text{weakly in } L^q\bigl([0,T],L^p_{\mathrm{loc}}(\Omega)\bigr)\text{ for any } q<2,\ p<\frac{2\gamma}{1+2\theta},
\end{gathered}
$$

$$(\rho^\varepsilon)^\theta(-\Delta)^{-1}\operatorname{div}\bigl[\varphi(r)\rho^\varepsilon \boldsymbol{u}^\varepsilon\bigr] \rightharpoonup \overline{\rho^\theta}(-\Delta)^{-1}\operatorname{div}\bigl[\varphi(r)\rho\boldsymbol{u}\bigr],$$
$$\text{weakly in } L^q\bigl([0,T], L^p_{\mathrm{loc}}(\Omega)\bigr) \text{ for any } q<\infty,\ p \leqslant \frac{2\gamma}{1+2\theta}, \tag{3.10}$$

where we have used

$$(\rho^\varepsilon)^\theta \rightharpoonup \overline{\rho^\theta} \quad \text{weak-}* \text{ in } L^\infty\bigl([0,T], \mathcal{L}^{\gamma/\theta}(\Omega)\bigr) \text{ and}$$
$$(\rho^\varepsilon)^\theta \to \overline{\rho^\theta} \quad \text{in } C^0\bigl([0,T], L^p_{\mathrm{loc}}(\Omega)-w\bigr) \text{ for any } 1<p\leqslant \frac{\gamma}{\theta},\ 0<\theta<\frac{\gamma}{2}, \tag{3.11}$$

which follows from (2.11)–(2.12) and Appendix C of [11] (also see $(3.4)_2$).

From (2.14) we get:

$$(-\Delta)^{-1}\partial_r\bigl[(\rho^\varepsilon)^\gamma\partial_r\varphi\bigr] \rightharpoonup (-\Delta)^{-1}\partial_r\bigl[\,\overline{\rho^\gamma}\partial_r\varphi\bigr] \quad \text{in } L^{(2\gamma-1)/\gamma}\bigl([0,T], W^{1,(2\gamma-1)/\gamma}_{\mathrm{loc}}(\Omega)\bigr). \tag{3.12}$$

On the other hand, the imbedding $L^{(4\gamma+2)/(4\gamma-1)}_{\mathrm{loc}}(\Omega) \hookrightarrow\hookrightarrow W^{-1,(2\gamma+1)/(\gamma-1)}_{\mathrm{loc}}(\Omega)$, together with (3.11) with $p=(4\gamma+2)/(4\gamma-1)$ and a contradiction argument, yields

$$(\rho^\varepsilon)^\theta \to \overline{\rho^\theta} \quad \text{in } C^0\bigl([0,T], W^{-1,(2\gamma+1)/(\gamma-1)}_{\mathrm{loc}}(\Omega)\bigr). \tag{3.13}$$

Consequently, from (3.11), (3.12) and Sobolev's imbedding theorem ($W^{1,(2\gamma-1)/\gamma}_{\mathrm{loc}} \hookrightarrow L^{2(2\gamma-1)}_{\mathrm{loc}}$) it follows that

$$(\rho^\varepsilon)^\theta(-\Delta)^{-1}\partial_r\bigl[(\rho^\varepsilon)^\gamma\partial_r\varphi\bigr] \rightharpoonup \overline{\rho^\theta}(-\Delta)^{-1}\partial_r\bigl[\,\overline{\rho^\gamma}\partial_r\varphi\bigr], \tag{3.14}$$

weakly in $L^{(2\gamma-1)/\gamma}([0,T], L^p_{\mathrm{loc}}(\Omega))$ with $p=\frac{2\gamma(2\gamma-1)}{\gamma+2\theta(2\gamma-1)}$ (>1 for $\theta<\gamma/2$).

If we make use of (2.11), (2.14) and $W^{1,(\beta+\theta)/\beta}_{\mathrm{loc}} \hookrightarrow L^{2(\beta+\theta)/(\beta-\theta)}_{\mathrm{loc}}$, we find for any $K \Subset \Omega$ and $0<\theta<\min\{\gamma-1,\gamma/2\}$ that

$$\begin{aligned}
&\varepsilon^\lambda\bigl\|(\rho^\varepsilon)^\theta(-\Delta)^{-1}\partial_r\bigl[(\rho^\varepsilon)^\beta\partial_r\varphi\bigr]\bigr\|_{L^1((0,T)\times K)}\\
&\quad\leqslant C\varepsilon^\lambda\bigl\|(\rho^\varepsilon)^\theta\bigr\|_{L^{2(\beta+\theta)/(\beta+3\theta)}((0,T)\times K)}\bigl\|(-\Delta)^{-1}\partial_r\bigl[(\rho^\varepsilon)^\beta\partial_r\varphi\bigr]\bigr\|_{W^{1,(\beta+\theta)/\beta}((0,T)\times K)}\\
&\quad\leqslant C\varepsilon^\lambda\|\rho^\varepsilon\|^\theta_{L^{2\theta(\beta+\theta)/(\beta+3\theta)}((0,T)\times K)}\bigl\|(\rho^\varepsilon)^\beta\bigr\|_{L^{(\beta+\theta)/\beta}((0,T)\times K)}\\
&\quad\leqslant C\varepsilon^{\lambda\theta/(\beta+\theta)}\|\rho^\varepsilon\|^\theta_{L^\gamma((0,T)\times K)}\bigl\|\varepsilon^{\lambda/(\beta+\theta)}\rho^\varepsilon\bigr\|^\beta_{L^{\beta+\theta}((0,T)\times K)} \to 0, \quad \text{and}\\
&\varepsilon^\lambda\bigl\|(\rho^\varepsilon)^{\theta+\beta}\bigr\|_{L^1((0,T)\times K)}\\
&\quad\leqslant C\varepsilon^{\lambda(\gamma-1-\theta)/(\beta+\gamma-1)}\bigl\|\varepsilon^{\lambda/(\beta+\gamma-1)}\rho^\varepsilon\bigr\|^{\beta+\theta}_{L^{\beta+\gamma-1}((0,T)\times K)} \to 0 \quad \text{as } \varepsilon\to 0.
\end{aligned} \tag{3.15}$$

From (3.5) we easily get:

$$(-\Delta)^{-1}\partial_r\left[r\rho^\varepsilon u_i^\varepsilon u_j^\varepsilon \partial_r\left(\frac{\varphi}{r}\right)\right] \rightharpoonup (-\Delta)^{-1}\partial_r\left[r\rho u_i u_j \partial_r\left(\frac{\varphi}{r}\right)\right], \tag{3.16}$$

weakly in $L^2([0,T], W^{1,p}_{\mathrm{loc}}(\Omega))$ for all $1 < p < 2\gamma/(\gamma+1)$. So, by virtue of (3.11), (3.13), (3.16) and Sobolev's imbedding theorem, we obtain:

$$(\rho^\varepsilon)^\theta(-\Delta)^{-1}\partial_r\left[r\rho^\varepsilon\left(u_1^\varepsilon\right)^2\partial_r\left(\frac{\varphi}{r}\right)\right] \rightharpoonup \overline{\rho^\theta}(-\Delta)^{-1}\partial_r\left[r\rho u_1^2\partial_r\left(\frac{\varphi}{r}\right)\right], \tag{3.17}$$

weakly in $L^2([0,T], L^p_{\mathrm{loc}}(\Omega))$ for any $1 < p < 2\gamma/(2\theta+1)$ (recall here $2\gamma/(2\theta+1) > 1$ for $\theta < \gamma/2$).

Analogously to (3.17), we can show that

$$(\rho^\varepsilon)^\theta(-\Delta)^{-1}\partial_z\left[r\rho^\varepsilon u_1^\varepsilon u_2^\varepsilon\partial_r\left(\frac{\varphi}{r}\right)\right] \rightharpoonup \overline{\rho^\theta}(-\Delta)^{-1}\partial_z\left[r\rho u_1 u_2\partial_r\left(\frac{\varphi}{r}\right)\right], \tag{3.18}$$

weakly in $L^2([0,T], L^p_{\mathrm{loc}}(\Omega))$ for any $1 < p < 2\gamma/(2\theta+1)$.

Finally, we estimate the commutator in (2.13). For $\psi \in C_0^\infty(\mathbb{R}^+\times\Omega)$, we have by the symmetry of the operator R_{ij} that

$$\begin{aligned}
&\int\limits_{\mathbb{R}^+\times\Omega} \psi(t,r,z)\left\{(\rho^\varepsilon)^\theta R_{ij}\left(\varphi\rho^\varepsilon u_i^\varepsilon u_j^\varepsilon\right) - (\rho^\varepsilon)^\theta u_i^\varepsilon R_{ij}\left(\varphi\rho^\varepsilon u_j^\varepsilon\right)\right\}\mathrm{d}r\,\mathrm{d}z\,\mathrm{d}t \\
&\quad = \int\limits_{\mathbb{R}^+\times\Omega} u_i^\varepsilon\left\{\varphi\rho^\varepsilon u_j^\varepsilon R_{ij}\left(\psi(\rho^\varepsilon)^\theta\right) - \psi(\rho^\varepsilon)^\theta R_{ij}\left(\varphi\rho^\varepsilon u_j^\varepsilon\right)\right\}\mathrm{d}r\,\mathrm{d}z\,\mathrm{d}t.
\end{aligned} \tag{3.19}$$

On the other hand, from Corollary 4.1 of [6], $(3.4)_2$ and $(3.11)_2$ one gets:

$$\begin{aligned}
&\varphi\rho^\varepsilon u_j^\varepsilon R_{ij}\left[\psi(\rho^\varepsilon)^\theta\right] - \psi(\rho^\varepsilon)^\theta R_{ij}\left(\varphi\rho^\varepsilon u_j^\varepsilon\right) \\
&\quad \to \varphi\rho u_j R_{ij}\left[\psi\overline{\rho^\theta}\right] - \psi\overline{\rho^\theta}R_{ij}(\varphi\rho u_j) \quad \text{in } C^0\left([0,T], \mathcal{L}^s_{\mathrm{loc}}(\Omega) - w\right)
\end{aligned} \tag{3.20}$$

for any $1 < s < 2\gamma/(\gamma+2\theta+1)$ with $\theta < (\gamma-1)/2$. Since $\mathcal{L}^s_{\mathrm{loc}}(\Omega) \hookrightarrow \mathcal{H}^{-1}_{\mathrm{loc}}(\Omega)$, the weak convergence in (3.20) is in fact strong convergence in $C^0([0,T], \mathcal{H}^{-1}_{\mathrm{loc}}(\Omega))$ (cf. (3.13)). Hence, combining (3.19) with (3.1) and (3.20), we conclude:

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int\limits_{\mathbb{R}^+\times\Omega} \psi\left\{(\rho^\varepsilon)^\theta R_{ij}\left(\varphi\rho^\varepsilon u_i^\varepsilon u_j^\varepsilon\right) - u_i^\varepsilon R_{ij}\left(\varphi\rho^\varepsilon u_j^\varepsilon\right)\right\}\mathrm{d}r\,\mathrm{d}z\,\mathrm{d}t \\
&\quad = \int\limits_{\mathbb{R}^+\times\Omega} \psi\left\{\overline{\rho^\theta}R_{ij}(\varphi\rho u_i u_j) - u_i R_{ij}\left(\varphi\rho u^j\right)\right\}\mathrm{d}r\,\mathrm{d}z\,\mathrm{d}t, \quad \psi \in C_0^\infty(\mathbb{R}^+\times\Omega).
\end{aligned} \tag{3.21}$$

Now, taking $\varepsilon \to 0$ in (2.13), using (2.14), (3.8), (3.10), and (3.14), (3.15), (3.17), (3.18) and (3.21), we obtain:

$$
\begin{aligned}
\varphi(r)\big(a\overline{\rho^{\gamma+\theta}} - \mu\mathcal{Q}\big) &= \partial_t\big\{\overline{\rho^\theta}(-\Delta)^{-1}\operatorname{div}(\varphi\rho\boldsymbol{u})\big\} + \overline{\rho^\theta}R_{ij}(\varphi\rho u_i u_j) \\
&\quad - \overline{\rho^\theta}u_i R_{ij}(\varphi\rho u_j) + \operatorname{div}\big\{\overline{\rho^\theta}\boldsymbol{u}(-\Delta)^{-1}\operatorname{div}[\varphi\rho\boldsymbol{u}]\big\} \\
&\quad + (-\Delta)^{-1}\operatorname{div}(\varphi\rho\boldsymbol{u})\left\{\theta\,\overline{\rho^\theta}\frac{u_1}{r} + (\theta-1)\mathcal{Q}\right\} - \overline{\rho^\theta}(-\Delta)^{-1}\partial_r\left\{r\rho u_1^2\partial_r\left[\frac{\varphi}{r}\right]\right\} \\
&\quad - \overline{\rho^\theta}(-\Delta)^{-1}\partial_z\left\{r\rho u_1u_2\partial_r\left[\frac{\varphi}{r}\right]\right\} - a\overline{\rho^\theta}(-\Delta)^{-1}\partial_r\big(\overline{\rho^\gamma}\partial_r\varphi\big) \\
&\quad + \mu\overline{\rho^\theta}(-\Delta)^{-1}\partial_r\left\{r\partial_r u_1\partial_r\left[\frac{\varphi}{r}\right]\right\} - \mu\overline{\rho^\theta}(-\Delta)^{-1}\big\{\partial_z^2 u_1\partial_r\varphi\big\} \\
&\quad + \mu\overline{\rho^\theta}(-\Delta)^{-1}\partial_{rz}^2(u_2\partial_r\varphi) + \mu\overline{\rho^\theta}(-\Delta)^{-1}\partial_z\left\{r\partial_r u_2\partial_r\left[\frac{\varphi}{r}\right]\right\} \\
&\quad + \mu\overline{\rho^\theta}(-\Delta)^{-1}\partial_r\left[\frac{u_1}{r^2}\varphi\right]
\end{aligned}
\tag{3.22}
$$

in the sense of distributions, where $0 < \theta < (\gamma-1)/2$ and $\mathcal{Q}$ denotes the weak limit of $\rho^\varepsilon\operatorname{div}\boldsymbol{u}^\varepsilon$.

Now, approximating ρ^θ by $b \in C^1(\mathbb{R})$ satisfying $|b(s)| \leqslant C$ and $|b'(s)s| \leqslant C$, one finds by (2.9) that

$$
\partial_t(\rho^\varepsilon)^\theta + \frac{1}{r}\partial_r\big[r(\rho^\varepsilon)^\theta u_1^\varepsilon\big] + \partial_z\big[(\rho^\varepsilon)^\theta u_2^\varepsilon\big] + (\theta-1)(\rho^\varepsilon)^\theta\left(\frac{u_1^\varepsilon}{r} + \operatorname{div}\boldsymbol{u}^\varepsilon\right) = 0.
$$

Thus, by applying Eqs. (3.8), (3.11), and letting $\varepsilon \to 0$ in the above equation, we have:

$$
\partial_t\overline{\rho^\theta} + \frac{1}{r}\partial_r\big(r\overline{\rho^\theta}u_1\big) + \partial_z\big(\overline{\rho^\theta}u_2\big) + (\theta-1)\overline{\rho^\theta}\frac{u_1}{r} + (\theta-1)\mathcal{Q} = 0. \tag{3.23}
$$

Therefore, using (3.7) and (3.23), following a similar procedure to the one used in [12, pp. 8–9], we deduce that

$$
\varphi(r)\overline{\rho^\theta}\big(a\overline{\rho^\gamma} - \mu\operatorname{div}\boldsymbol{u}\big) = \text{R.H.S. of (3.22)},
$$

which combined with (3.22) proves:

Lemma 3.1. *$a\,\overline{\rho^{\gamma+\theta}} - \mu\mathcal{Q} = \overline{\rho^\theta}(a\overline{\rho^\gamma} - \mu\operatorname{div}\boldsymbol{u})$ for all $0 < \theta < \frac{\gamma-1}{2}$.*

As a consequence of Lemma 3.1, one has

$$
\mu\big(\mathcal{Q} - \overline{\rho^\theta}\operatorname{div}\boldsymbol{u}\big) = a\overline{\rho^{\gamma+\theta}} - a\overline{\rho^\theta}\,\overline{\rho^\gamma}. \tag{3.24}
$$

Notice that by convexity,

$$\overline{\rho^{\gamma}} \leqslant \overline{\rho^{\gamma+\theta}}^{\frac{\gamma}{\gamma+\theta}}, \qquad \overline{\rho^{\theta}} \leqslant \overline{\rho^{\gamma+\theta}}^{\frac{\theta}{\gamma+\theta}},$$

which together with (3.24) yields

$$\mathcal{Q} \geqslant \overline{\rho^{\theta}} \operatorname{div} \boldsymbol{u}. \tag{3.25}$$

To exclude possible concentration on the line $r = 0$, we will use the following estimate which is a similar version of the key Lemma 3.2 of [10]:

Lemma 3.2. *Let* $0 < \theta < \min\{1/2, (\gamma - 1)/2\}$ *and* $\frac{1}{2}(1 - \theta + \sqrt{1 + 6\theta + \theta^2}\,) \leqslant \gamma$. *Then*

$$\rho^{\theta} - \overline{\rho}^{\theta} \in L^{2/\theta}\big([0, T], \mathcal{L}^{2/\theta}(\mathbb{R}^+ \times \mathbb{R})\big).$$

Proof. Using the estimates (2.8) and (2.11), this lemma can be shown by the same proof as that of Lemma 3.2 in [10], and hence we omit its proof here. $\square$

Now, we are able to complete the proof of the precompactness. First we recall that $(\rho, \boldsymbol{u})$ satisfies (1.5) (cf. the paragraph below (3.7)). Thus, by approximating ρ^{θ} by $b(\rho)$ again, we find by (1.5) that

$$\partial_t \rho^{\theta} + \frac{1}{r}\partial_r\big(r\rho^{\theta}u_1\big) + \partial_z\big(\rho^{\theta}u_2\big) + (\theta - 1)\rho^{\theta}\left(\frac{u_1}{r} + \operatorname{div}\boldsymbol{u}\right) = 0. \tag{3.26}$$

Subtracting (3.23) from (3.26) and utilizing (3.25), we arrive at

$$\begin{aligned}
&\partial_t\big(\rho^{\theta} - \overline{\rho^{\theta}}\,\big) + \frac{1}{r}\partial_r\big[ru_1\big(\rho^{\theta} - \overline{\rho^{\theta}}\,\big)\big] + \partial_z\big[u_2\big(\rho^{\theta} - \overline{\rho^{\theta}}\,\big)\big] + (\theta - 1)\frac{u_1}{r}\big(\rho^{\theta} - \overline{\rho^{\theta}}\,\big) \\
&\quad = (1 - \theta)\big(\rho^{\theta} \operatorname{div}\boldsymbol{u} - \mathcal{Q}\big) \\
&\quad \leqslant (1 - \theta)\big(\rho^{\theta} - \overline{\rho^{\theta}}\,\big)\operatorname{div}\boldsymbol{u}, \quad 0 < \theta < \min\left\{1, \frac{\gamma - 1}{2}\right\}.
\end{aligned} \tag{3.27}$$

Now set $f := r^{\theta}(\rho^{\theta} - \overline{\rho^{\theta}})$, then multiplying (3.27) by r^{θ} and mollifying the resulting inequality, we obtain:

$$\partial_t f_{\delta} + \frac{1}{r^{1-\theta}}\partial_r\big(r^{1-\theta}u_1 f_{\delta}\big) + \partial_z(u_2 f_{\delta}) \leqslant (1 - \theta)\frac{u_1}{r} f_{\delta} + (1 - \theta) f_{\delta} \operatorname{div}\boldsymbol{u} + h_{\delta}, \tag{3.28}$$

where $f_{\delta} := f * j_{\delta}$ and h_{δ} satisfies $h_{\delta} \to 0$ in $L^1_{\mathrm{loc}}(\mathbb{R}^+ \times \Omega)$ by applying Lemma 2.3 of [11, p. 43].

Keeping in mind that $f \geqslant 0$ by convexity, we multiply (3.28) by $\frac{1}{\theta}(f_{\delta} + \eta)^{1/\theta - 1}$ $(\eta > 0)$, take $\delta \to 0$ and then $\eta \to 0$ to deduce that

$$\partial_t f^{1/\theta} + \operatorname{div}\big(\boldsymbol{u} f^{1/\theta}\big) \leqslant 0 \quad \text{in } \mathcal{D}'\big((0, T) \times \Omega\big) \tag{3.29}$$

for $0 < \theta < \min\{1/2, (\gamma - 1)/2\}$.

Next, we modify the proof of the last part in [10] to show that $(\rho, \boldsymbol{u})$ is indeed a finite energy weak solution to (1.1). First observe that by (3.11) and (2.6),

$$\overline{\rho^\theta}(t,r,z) \rightharpoonup \overline{\rho^\theta}(0,r,z) = \rho_0^\theta(r,z) \quad \text{weakly in } L^{1/\theta}_{\text{loc}}(\Omega) \text{ as } t \to 0. \tag{3.30}$$

On the other hand, in view of convexity and (3.2), we see that for any $K \Subset \Omega$,

$$\int_K \rho_0 r\,\mathrm{d}r\,\mathrm{d}z \leqslant \varliminf_{t\to 0} \int_K \overline{\rho^\theta}^{1/\theta}(t,r,z) r\,\mathrm{d}r\,\mathrm{d}z \leqslant \varlimsup_{t\to 0} \int_K \left(\rho^\theta\right)^{1/\theta}(t,r,z) r\,\mathrm{d}r\,\mathrm{d}z = \int_K \rho_0 r\,\mathrm{d}r\,\mathrm{d}z,$$

from which it follows that $\lim_{t\to 0} \int_K \overline{\rho^\theta}^{1/\theta}(t,r,z) r\,\mathrm{d}r\,\mathrm{d}z = \int_K (\rho_0^\theta)^{1/\theta} r\,\mathrm{d}r\,\mathrm{d}z$. This together with (3.30) yields:

$$\lim_{t\to 0} \int_K \left(\overline{\rho^\theta}(t,r,z) - \rho_0^\theta(r,z)\right)^{1/\theta} r\,\mathrm{d}r\,\mathrm{d}z = 0. \tag{3.31}$$

In the same manner, we obtain $\lim_{t\to 0} \int_K (\rho^\theta(t,r,z) - \rho_0^\theta(r,z))^{1/\theta} r\,\mathrm{d}r\,\mathrm{d}z = 0$, which combined with (3.31) gives that for any $K \Subset \Omega$,

$$\lim_{t\to 0} \int_K \left(\rho^\theta - \overline{\rho^\theta}\right)^{1/\theta}(t,r,z) r\,\mathrm{d}r\,\mathrm{d}z = 0. \tag{3.32}$$

Let $\psi_\varepsilon(z) \in C_0^\infty(\mathbb{R})$, $\varphi_\varepsilon(r), \xi_\delta^\tau(t) \in C_0^\infty(\mathbb{R}^+)$ with

$$\psi_\varepsilon(z) := \begin{cases} 0, & |z| \geqslant \dfrac{1}{\varepsilon} + 1, \\ 1, & |z| \leqslant \dfrac{1}{\varepsilon}, \end{cases}$$

$$\varphi_\varepsilon(r) := \begin{cases} 0, & r \leqslant \dfrac{\varepsilon}{2}, \\ 1, & \varepsilon \leqslant r \leqslant \dfrac{1}{\varepsilon}, \\ 0, & r \geqslant \dfrac{1}{\varepsilon} + 1, \end{cases} \qquad \xi_\delta^\tau(t) := \begin{cases} 0, & t \leqslant \dfrac{\delta}{2}, \\ 1, & \delta \leqslant t \leqslant \dfrac{\tau}{2}, \\ 0, & t \geqslant \dfrac{3\tau}{2}, \end{cases}$$

$$\left|\partial_r \varphi_\varepsilon(r)\right| \leqslant \frac{C}{\varepsilon} \quad \text{for } r \leqslant \varepsilon, \qquad \partial_t \xi_\delta^\tau(t) \leqslant 0 \quad \text{for } \frac{\tau}{2} \leqslant t \leqslant \frac{3\tau}{2} \quad \text{and}$$

$$\partial_t \xi_\delta^\tau(t) \leqslant -C_1 \tau \quad \text{for all } \frac{3\tau}{4} \leqslant t \leqslant \frac{5\tau}{4} \text{ and some constant } C_1 > 0,$$

where $\tau \geqslant 2\delta$ is an arbitrary but fixed number. Then, multiplication of (3.29) with $\varphi_\varepsilon(r)\psi_\varepsilon(z)\xi_\delta^\tau(t)$ and integration over $\mathbb{R}^+ \times \Omega$ result in:

$$
\begin{aligned}
&C_1\tau \int_{3\tau/4}^{5\tau/4}\int_{\Omega} f^{1/\theta}(t,r,z)\varphi_\varepsilon(r)\psi_\varepsilon(z)\,\mathrm{d}r\,\mathrm{d}z\,\mathrm{d}t \\
&\quad \leqslant \frac{C}{\delta}\int_0^{\delta}\int_{\Omega} f^{1/\theta}(t,r,z)\varphi_\varepsilon(r)\psi_\varepsilon(z)\,\mathrm{d}r\,\mathrm{d}z\,\mathrm{d}t \\
&\qquad + \frac{C}{\varepsilon}\int_0^{2\tau}\int_0^{\varepsilon}\int_{\mathbb{R}} |u_1| f^{1/\theta}\,\mathrm{d}r\,\mathrm{d}z\,\mathrm{d}t + C\int_0^{\tau}\int_{\{r\geqslant\varepsilon, z\in\mathbb{R}\}\cup\{r\in\mathbb{R}^+, |z|\geqslant\varepsilon\}} |\boldsymbol{u}| f^{1/\theta}\,\mathrm{d}r\,\mathrm{d}z\,\mathrm{d}t. \quad (3.33)
\end{aligned}
$$

Taking into account that $|\boldsymbol{u} f^{1/\theta}| \leqslant C\rho|\boldsymbol{u}|$, we infer by Lemma 3.2, (2.8) and (2.11) that

$$
f^{1/\theta},\ \frac{u_1}{r} f^{1/\theta},\ \boldsymbol{u} f^{1/\theta} \in L^1\big((0,T)\times\Omega\big).
$$

Hence, letting first $\delta \to 0$ and then $\varepsilon \to 0$ in (3.33), employing (3.32), we conclude $f^{1/\theta} = 0$, a.e. for $(t,r,z) \in (3\tau/4, 5\tau/4)\times\Omega$. Since τ is arbitrary, this implies:

$$
\rho^\theta(t,r,z) = \overline{\rho^\theta}(t,r,z), \quad \text{a.e. } (t,r,z) \in \mathbb{R}^+\times\mathbb{R}^+\times\mathbb{R}
$$

for all $\theta > 0$ satisfying the conditions of Lemma 3.2. By virtue of convexity, the Young measure associated with $\{\rho_\varepsilon(t,r,z)\}$ is the Dirac mass, thus,

$$
\rho^\varepsilon \to \rho \quad \text{strongly in } L^p_{\mathrm{loc}}(\mathbb{R}^+\times\mathbb{R}^+\times\mathbb{R})\ \forall p < 2\gamma - 1. \tag{3.34}
$$

To complete the proof of Theorem 1.1, it remains to prove (1.6)–(1.8) and (1.10). To this end, making use of (3.7) and (3.34), we can show, in the same manner as in the derivation of (3.29)–(3.33) in [10], that $(\rho, \boldsymbol{u})$ obtained in (3.1) and (3.2) satisfies (1.6), (1.7).

To show (1.8), we shall use concentration compactness arguments similar to those of Lions for the stationary isothermal case and the estimate (2.16). First, with the help of (2.11) we find that for $\phi \in C_0^1(\mathbb{R}^3)$ with $\phi_r(t,0,z) = 0$,

$$
\left|\int_0^T\int_0^h\int_{\mathbb{R}} \rho^\varepsilon u_1^\varepsilon u_2^\varepsilon \phi_r r\,\mathrm{d}r\,\mathrm{d}z\,\mathrm{d}t\right| \leqslant C \sup_{r\in[0,h],\, t\in[0,T],\, z\in\mathbb{R}} \big|\phi_r(t,r,z)\big| \to 0 \quad \text{as } h \to 0,
$$

which together with (3.5) implies:

$$
\int_0^T\int_\varepsilon^\infty\int_{\mathbb{R}} \rho^\varepsilon u_1^\varepsilon u_2^\varepsilon \phi_r r\,\mathrm{d}r\,\mathrm{d}z\,\mathrm{d}t \to \int_0^T\int_{\mathbb{R}^+\times\mathbb{R}} \rho u_1 u_2 \phi_r r\,\mathrm{d}r\,\mathrm{d}z\,\mathrm{d}t \quad \text{as } \varepsilon \to 0 \tag{3.35}
$$

for any $\phi \in C_0^1(\mathbb{R}^3)$ with $\phi_r(t,0,z)=0$. If we employ (2.16), we easily see that

$$\int_0^T \int_0^h \int_{\mathbb{R}} \left\{a(\rho^\varepsilon)^\gamma + \varepsilon^\lambda (\rho^\varepsilon)^\beta\right\}\phi_z r\,\mathrm{d}r\,\mathrm{d}z\,\mathrm{d}t \leqslant Ch^{1/3} \to 0 \quad \text{as } h \to 0.$$

Therefore, utilizing (3.6) and (3.34), one gets analogously to (3.35) that for $\phi \in C_0^1(\mathbb{R}^3)$,

$$\int_0^T \int_\varepsilon^\infty \int_{\mathbb{R}} \left\{a(\rho^\varepsilon)^\gamma + \varepsilon^\lambda (\rho^\varepsilon)^\beta\right\}\phi_z r\,\mathrm{d}r\,\mathrm{d}z\,\mathrm{d}t \to \int_0^T \int_{\mathbb{R}^+\times\mathbb{R}} a\rho^\gamma \phi_z r\,\mathrm{d}r\,\mathrm{d}z\,\mathrm{d}t \quad \text{as } \varepsilon \to 0. \tag{3.36}$$

Now, for any $T>0$, $\rho^\varepsilon (u_2^\varepsilon)^2 r$ is uniformly bounded with respect to ε in $L^1((0,T)\times\mathbb{R}^+\times\mathbb{R})$ by virtue of (2.11). Therefore, $\rho^\varepsilon (u_2^\varepsilon)^2 r\,\mathrm{d}r\,\mathrm{d}z\,\mathrm{d}t$ converges weakly in the sense of measures (on $[0,T]\times\mathbb{R}_0^+\times\mathbb{R}$) to a bounded non-negative Radon measure $\nu(t,r,z)$:

$$\rho^\varepsilon \left(u_2^\varepsilon\right)^2 r\,\mathrm{d}r\,\mathrm{d}z\,\mathrm{d}t \rightharpoonup \nu \quad \text{in the sense of measures.} \tag{3.37}$$

On the other hand, since ν is bounded, the set $\{(t,r,z) \mid \nu(\{(t,r,z)\})>0\}$ is at most countable (also cf. [3, p. 13]). Hence, by the Lebesgue decomposition and the Radon–Nikodym theorem, there is a $f \in L^1$, an at most countable set J (possibly empty), distinct points $\{t_i,r_i,z_i\}_{i\in J} \in [0,T]\times\mathbb{R}_0^+\times\mathbb{R}$ and positive constants $\{c_i\}_{i\in J}$, such that

$$\nu = fr\,\mathrm{d}r\,\mathrm{d}z\,\mathrm{d}t + \sum_{i\in J} c_i \delta(t_i,r_i,z_i), \quad \sum_{i\in J} c_i < \infty. \tag{3.38}$$

Moreover, by virtue of (3.5), we easily see that

$$f = \rho(u_2)^2, \quad r_i = 0. \tag{3.39}$$

Thus, for $\phi \in C_0^1([0,T]\times\mathbb{R}_0^+\times\mathbb{R})$ with $\phi_r(t,0,z)=0$, we take $\phi_\varepsilon \in C_0^1([0,T]\times\mathbb{R}_0^+\times\mathbb{R})$ with $\partial_r\phi_\varepsilon(t,\varepsilon,z)=0$ and $\phi_\varepsilon \to \phi$ in C^1. Recalling that test functions in the weak form of (2.3) are those in C_0^1 with the first order derivative in r vanishing at $r=\varepsilon$, then we test (2.3) with $r\phi_\varepsilon$, then take $\varepsilon \to 0$ and make use of (3.1), (3.4) and (3.35)–(3.39) to deduce that

$$\begin{aligned}
&\int_{\mathbb{R}^+\times\mathbb{R}} \rho u_2 \phi|_{t_1}^{t_2} r\,\mathrm{d}r\,\mathrm{d}z - \int_{t_1}^{t_2} \int_{\mathbb{R}^+\times\mathbb{R}} \left\{\rho u_2 \phi_t + \rho u_1 u_2 \phi_r + \rho(u_2)^2 \phi_z\right\} r\,\mathrm{d}r\,\mathrm{d}z\,\mathrm{d}t \\
&\quad = \int_{t_1}^{t_2} \int_{\mathbb{R}^+\times\mathbb{R}} \left\{a\rho^\gamma \phi_z - \mu \partial_r u_2 \phi_r - \partial_z u_2 \phi_z\right\} r\,\mathrm{d}r\,\mathrm{d}z\,\mathrm{d}t + \sum_{i\in J} c_i \phi_z(t_i,0,z_i)
\end{aligned} \tag{3.40}$$

for any $0 \leqslant t_1 \leqslant t_2 \leqslant T$, $\phi \in C_0^1([0,T]\times\mathbb{R}_0^+\times\mathbb{R})$ with $\phi_r(t,0,z)=0$.

We are going to show from (3.40) that $c_i = 0$ or in other words that $J = \emptyset$. We argue by contradiction and assume $c_l \neq 0$ for some $l \in J$. Let $\chi(r) \in C_0^\infty(\mathbb{R})$ such that $\chi(r) = 1$ for $|r| \leqslant 1/2$ and $\chi(r) = 0$ for $|r| \geqslant 1$. If we take $\phi = \chi((t - t_l)/\delta)\chi(r)(z - z_l)\chi((z - z_l)/\delta)$ (δ small) in (3.40) and keep in mind that $\sum_{i\in J} c_i < \infty$, we obtain:

$$c_l \leqslant C\delta + \int_0^T \int_{\mathbb{R}^+} \int_{z_l-\delta}^{z_l+\delta} \left\{\rho|u_2| + \rho(u_2)^2 + \rho^\gamma + |\partial_z u_2|\right\} r \,\mathrm{d}r \,\mathrm{d}z \,\mathrm{d}t$$
$$+ \sum_{i\in J,\, i\neq l,\, |t_i-t_l|+|z_i-z_l|\leqslant 2\delta} c_i \to 0 \quad \text{as } \delta \to 0.$$

Hence, $J = \emptyset$ and (1.8) holds. Thus, the limit $(\rho, \boldsymbol{u})$ is indeed a finite energy weak solutions of (1.1)–(1.3).

Finally, for $T, L, h > 0$ let $\xi \in C_0^1(\mathbb{R}^3)$, $\chi^h \in C_0^\infty(\mathbb{R})$ satisfying $\xi(t, r, z) = 1$ when $(t, r, z) \in [0, T] \times [0, 1] \times [-L, L]$, $\chi^h(r) = 0$ when $0 \leqslant r \leqslant h$ and $\chi^h(r) = 1$ when $r \geqslant 2h$. Then, taking $\phi = r^{1+\alpha}\xi(t, r, z)\chi^h(r)$ $(\alpha > 0)$ in (1.8), we obtain (1.10) by the same arguments as used for (2.16). This completes the proof of Theorem 1.1.

4. Existence of the approximate solutions

In this section we prove the existence of solutions of (2.1)–(2.5) by adapting the ideas in [12, Theorem 7.2] and [5,6]. The (approximate) solutions will be constructed by means of a three-level approximation scheme based on a modified system of (2.1)–(2.3):

$$\partial_t\rho + \frac{1}{r}\partial_r(r\rho u_1) + \partial_z(\rho u_2) = \delta\left[\frac{1}{r}\partial_r(r\partial_r\rho) + \partial_z^2\rho\right], \tag{4.1}$$

$$\begin{aligned}&\partial_t(\rho u_1) + \frac{1}{r}\partial_r\left[r\rho u_1^2\right] + \partial_z[\rho u_1 u_2] + \delta\nabla u_1\cdot\nabla\rho\\ &\quad = -a\partial_r\rho^\gamma - \varepsilon^\lambda\partial_r\rho^\beta + \mu\left[\frac{1}{r}\partial_r(r\partial_r u_1) + \partial_z^2 u_1\right] - \mu\frac{u_1}{r^2},\end{aligned} \tag{4.2}$$

$$\begin{aligned}&\partial_t(\rho u_2) + \frac{1}{r}\partial_r(r\rho u_1 u_2) + \partial_z\left[\rho u_2^2\right] + \delta\nabla u_2\cdot\nabla\rho\\ &\quad = -a\partial_z\rho^\gamma - \varepsilon^\lambda\partial_z\rho^\beta + \mu\left[\frac{1}{r}\partial_r(r\partial_r u_2) + \partial_z^2 u_2\right],\end{aligned} \tag{4.3}$$

where $\varepsilon, \delta, \beta, \lambda > 0$ are constants, ε and δ are small.

The first step is to solve the problem (4.1)–(4.3) in the square domain $\Omega_R := (\varepsilon, R) \times (R, R)$, then the second step consists of letting the artificial viscosity δ go to zero to obtain a solution of (2.1)–(2.5) on the domain Ω_R, and finally, in the third step we prove the existence of the solutions to (2.1)–(2.5) by passing to the limit letting $R \to \infty$.

Step 1. *The first level approximate solutions*. We consider the system (4.1)–(4.3) in $\Omega_R := (\varepsilon, R) \times (R, R)$, together with initial and boundary conditions:

$$\rho(0, r, z) = \rho_0, \qquad \rho\boldsymbol{u}(0, r, z) = \boldsymbol{m}_0, \quad (r, z) \in \Omega_R, \tag{4.4}$$

$$\nabla\rho \cdot \boldsymbol{n} = 0, \qquad \boldsymbol{u} \cdot \boldsymbol{n} = 0, \qquad \operatorname{curl} \boldsymbol{u} = 0 \quad \text{on } \partial\Omega_R,\ t \geqslant 0, \tag{4.5}$$

where $\boldsymbol{n}$ is the outer normal vector to Ω_R. We have:

Lemma 4.1. *Let $\beta > \max\{4, \gamma\}$. Assume $\rho_0 \in L^\gamma(\Omega_R) \cap L^\beta(\Omega_R) \cap L^\infty(\Omega_R)$, $\inf_{\Omega_R} \rho_0 > 0$ and $\boldsymbol{m}_0/\rho_0 \in L^2(\Omega_R)$. Then there is a global weak solution $(\rho, \boldsymbol{u})$ of* (4.1)–(4.5) *with $\rho \geqslant 0$ a.e., such that $\rho \in L^{\beta+1}((0, T) \times \Omega_R)$ of* (4.1)–(4.5), *such that*

$$\sup_{t\in[0,T]} \left(\|\rho(t)\|^\gamma_{L^\gamma(\Omega_R)} + \varepsilon^\lambda \|\rho(t)\|^\beta_{L^\beta(\Omega_R)} + \left\| \left(\sqrt{\rho}\,\boldsymbol{u}\right)(t) \right\|^2_{L^2(\Omega_R)} \right) + \int_0^T \left(\|\boldsymbol{u}\|^2_{L^2(\Omega_R)} + \|\nabla\boldsymbol{u}\|^2_{L^2(\Omega_R)} \right)(t)\, \mathrm{d}t \leqslant C, \tag{4.6}$$

$$\delta \int_0^T \|\nabla\rho(t)\|^2_{L^2(\Omega_R)}\, \mathrm{d}t \leqslant C, \tag{4.7}$$

where the constant C does not depend on δ but on ε, R, β, ρ_0 and $\boldsymbol{m}_0$. Moreover, the energy inequality

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega_R} \left(\frac{\rho|\boldsymbol{u}|^2}{2} + \frac{a\rho^\gamma}{\gamma - 1} + \frac{\varepsilon^\lambda \rho^\beta}{\beta - 1} \right) r\, \mathrm{d}r\, \mathrm{d}z + \mu \int_{\Omega_R} \left(|\nabla\boldsymbol{u}|^2 + \frac{u_1^2}{r^2} \right) r\, \mathrm{d}r\, \mathrm{d}z \leqslant 0 \tag{4.8}$$

holds in $\mathcal{D}'(0, T)$.

Proof. First notice that for the domain Ω_R, the boundary conditions (4.5) imply:

$$\begin{aligned} &u_1 = \partial_r u_2 = 0 \quad \text{on } \{\varepsilon\} \times (-R, R) \cup \{R\} \times (-R, R), \\ &u_2 = \partial_z u_1 = 0 \quad \text{on } (\varepsilon, R) \times \{-R\} \cup (\varepsilon, R) \times \{R\}. \end{aligned} \tag{4.9}$$

Thus, multiplying (4.1), (4.2) and (4.3) by $r\rho$, ru_1 and ru_2 respectively and integrating the resulting equations, then integrating by parts, using (4.5) and (4.9), we obtain (4.8). Consequently, (4.6) follows from (4.8) and the (generalized) Poincaré inequality.

Using (4.7) and following the same procedure as in the proof of Proposition 3.1 in [4] and Proposition 2.1 in [5], we can obtain the existence of weak solutions to (4.1)–(4.5) and the estimate (4.7) by solving (4.1) respectively (4.2)–(4.3) directly respectively by a Faedo–Galerkin approximation. The proof of Lemma 4.1 is complete. □

Step 2. *The vanishing artificial viscosity limit*. In this step we want to pass to the limit in (4.1)–(4.3) letting $\delta \to 0$. Accordingly, consider the problem (2.1)–(2.3) in the domain Ω_R with initial and boundary conditions:

$$\begin{aligned} &\rho(0,r,z) = \rho_0, \qquad \rho\boldsymbol{u}(0,r,z) = \boldsymbol{m}_0, \quad (r,z) \in \Omega_R, \\ &\boldsymbol{u}\cdot\boldsymbol{n} = 0, \qquad \operatorname{curl}\boldsymbol{u} = 0 \quad \text{on } \partial\Omega_R,\ t \geqslant 0. \end{aligned} \tag{4.10}$$

Then, we have:

Lemma 4.2. *Let* $\beta > \max\{4,\gamma\}$. *Assume* $\rho_0 \in L^\gamma(\Omega_R) \cap L^\beta(\Omega_R)$, $\rho_0 \geqslant 0$ *a.e. and* $\boldsymbol{m}_0/\sqrt{\rho_0} \in L^2(\Omega_R)$. *Then there is a global weak solution* $(\rho,\boldsymbol{u})$ *of* (2.1)–(2.3), (4.10) *with* $\rho \geqslant 0$ *a.e., such that for any* $T > 0$,

$$\begin{aligned} &\sup_{t\in[0,T]} \int_{\Omega_R} \left(\frac{\rho|\boldsymbol{u}|^2}{2} + \frac{a\rho^\gamma}{\gamma-1} + \frac{\varepsilon^\lambda\rho^\beta}{\beta-1}\right) r\,\mathrm{d}r\,\mathrm{d}z + \mu\int_0^T\int_{\Omega_R}\left(|\nabla\boldsymbol{u}|^2 + \frac{u_1^2}{r^2}\right) r\,\mathrm{d}r\,\mathrm{d}z\,\mathrm{d}t \\ &\quad \leqslant E_R(\rho_0,\boldsymbol{m}_0), \end{aligned} \tag{4.11}$$

where

$$E_R(\rho_0,\boldsymbol{m}_0) := \int_{\Omega_R}\left(\frac{|\boldsymbol{m}_0|^2}{2\rho_0} + \frac{a\rho_0^\gamma}{\gamma-1} + \frac{\varepsilon^\lambda\rho_0^\beta}{\beta-1}\right) r\,\mathrm{d}r\,\mathrm{d}z.$$

Proof. It is easy to find a smooth function sequence ρ_0^δ such that

$$\begin{aligned} &0 < C_1(\delta) \leqslant \rho_0^\delta(x) \leqslant C_2(\delta), \qquad \nabla\rho_0^\delta\cdot\boldsymbol{n}|_{\partial\Omega_R} = 0, \\ &\rho_0^\delta \to \rho_0 \quad \text{in } L^\gamma(\Omega_R)\cap L^\beta(\Omega_R) \text{ as } \delta\to 0. \end{aligned}$$

Take $\chi_1^\delta, \chi_2^\delta \in C_0^\infty(\mathbb{R})$ such that $\chi_1^\delta(r) = 1$ when $\varepsilon + 2\delta \leqslant r \leqslant R - 2\delta$, $\chi_2^\delta(z) = 1$ when $|z| \leqslant R - 2\delta$, and $\chi_1^\delta(r) = 0$ when $r \leqslant \varepsilon + \delta$ or $r \geqslant R - \delta$, $\chi_2^\delta(z) = 1$ when $|z| \geqslant R - \delta$. Set

$$\boldsymbol{m}_0^\delta(r,z) := \chi_1^\delta(r)\chi_2^\delta(z) \times \begin{cases} \left(\boldsymbol{m}_0/\sqrt{\rho_0}\right) * j_\delta\sqrt{\rho_0^\delta}, & \text{if } \rho_0(x) > 0, \\ 0, & \text{if } \rho_0(x) = 0. \end{cases}$$

Then, $\boldsymbol{m}_0^\delta/\sqrt{\rho_0^\delta} \to \boldsymbol{m}_0/\sqrt{\rho_0}$ in $L^2(\Omega_R)$.

We denote by $(\rho^\delta,\boldsymbol{u}^\delta)$ the solution of (4.1)–(4.6) with the initial data $(\rho_0^\delta,\boldsymbol{m}_0^\delta)$ obtained in Lemma 4.1. We first observe that as $\delta \to 0$,

$$\begin{aligned} &\delta\Delta\rho^\delta \to 0 \quad \text{in } L^2\big((0,T),H^{-1}(\Omega_R)\big), \\ &\delta\nabla u_j^\delta\cdot\nabla\rho^\delta \to 0 \quad \text{in } L^1\big((0,T)\times\Omega_R\big),\ j = 1,2, \end{aligned} \tag{4.12}$$

which easily follows from (4.6) and (4.7). Using (4.6) and (4.7), we get in the same manner as in the derivation of (2.14) (with $\varphi(r) \equiv 1$) that

$$\int_0^T \int_{\Omega_R} \{(\rho^\delta)^{\gamma+\theta} + \varepsilon^\lambda (\rho^\delta)^{\beta+\theta}\} r \,\mathrm{d}r \,\mathrm{d}z \,\mathrm{d}t \leqslant C, \quad \theta = \gamma - 1, \tag{4.13}$$

where C is a positive constant independent of δ.

The estimates (4.6) and (4.7) imply that $(\rho^\delta, \boldsymbol{u}^\delta) \rightharpoonup (\rho, \boldsymbol{u})$ weakly or weak-$*$. Now, letting $\delta \to 0$ in (4.1)–(4.3), employing (4.12), (4.6)–(4.7) and (4.13), we deduce by the same arguments as in Section 3 to prove precompactness that the weak limit $(\rho, \boldsymbol{u})$ is indeed a weak solution of (2.1)–(2.3), (4.10) on $[0, \infty) \times \Omega_R$. Moreover, the estimate (4.11) follows easily from (4.8), the lower semicontinuity of weak convergence and the convergence of $(\rho_0^\delta, \boldsymbol{m}_0^\delta)$.

It should be noticed here that Lemma 3.2 is in fact not needed in the above-mentioned limit process, because $\rho^\delta \in L^\infty((0,T), L^\beta(\Omega_R))$ $(\beta > 4)$ immediately implies $\rho^\theta, \overline{\rho^\theta} \in L^{2/\theta}((0,T), \mathcal{L}^{2/\theta}(\Omega_R))$. $\square$

Step 3. *Passing to the limit as $R \to \infty$.* In this step we pass to the limit as $R \to \infty$ in (2.1)–(2.3) and (4.10) to obtain a solution of (2.1)–(2.5). The main result of this section is the following:

Theorem 4.3. *Let $\beta > \max\{4, \gamma\}$ and denote $G_\varepsilon := (\varepsilon, \infty) \times \mathbb{R}$. Assume $\rho_0 \in \mathcal{L}^\gamma(G_\varepsilon) \cap \mathcal{L}^\beta(G_\varepsilon) \cap \mathcal{L}^1(G_\varepsilon)$, $\rho_0 \geqslant 0$ a.e. and $\boldsymbol{m}_0/\sqrt{\rho_0} \in \mathcal{L}^2(G_\varepsilon)$. Then there is a global weak solution $(\rho, \boldsymbol{u})$ of* (2.1)–(2.5) *with $(\rho_0^\varepsilon, \boldsymbol{m}_0^\varepsilon)$ replaced by $(\rho_0, \boldsymbol{m}_0)$, such that for any $T > 0$,*

$$\sup_{t\in[0,T]} \int_{G_\varepsilon} \left(\frac{\rho|\boldsymbol{u}|^2}{2} + \frac{a\rho^\gamma}{\gamma-1} + \frac{\varepsilon^\lambda \rho^\beta}{\beta-1}\right) r \,\mathrm{d}r \,\mathrm{d}z + \mu \int_0^T \int_{G_\varepsilon} \left(|\nabla \boldsymbol{u}|^2 + \frac{u_1^2}{r^2}\right) r \,\mathrm{d}r \,\mathrm{d}z \,\mathrm{d}t$$
$$\leqslant \int_{G_\varepsilon} \left(\frac{|\boldsymbol{m}_0|^2}{2\rho_0} + \frac{a\rho_0^\gamma}{\gamma-1} + \frac{\varepsilon^\lambda \rho_0^\beta}{\beta-1}\right) r \,\mathrm{d}r \,\mathrm{d}z, \tag{4.14}$$

$$\sup_{t\in[0,T]} \int_{G_\varepsilon} \rho(t,r,z) r \,\mathrm{d}r \,\mathrm{d}z \leqslant \int_{G_\varepsilon} \rho_0(r,z) r \,\mathrm{d}r \,\mathrm{d}z. \tag{4.15}$$

Proof. We first construct the approximation of $(\rho_0, \boldsymbol{m}_0)$ as follows:

$$\rho_0^R(r,z) := \rho_0(r,z), \qquad \boldsymbol{m}_0^R(r,z) := \boldsymbol{m}_0(r,z)\chi_1^R(r)\chi_2^R(z),$$

where $\chi_1^R, \chi_2^R \in C_0^\infty(\mathbb{R})$ satisfying $\chi_1^R(r) = 1$ when $\varepsilon + 1/R \leqslant r \leqslant R - 1$, $\chi_2^R(z) = 1$ when $|z| \leqslant R - 1$, and $\chi_1^R(r) = 0$ when $r \leqslant \varepsilon + 1/(2R)$ or $r \geqslant R - 1/2$, $\chi_2^R(z) = 1$ when $|z| \geqslant R - 1/2$. Then, it is easy to see that as $R \to \infty$,

$$\rho_0^R \to \rho_0 \quad \text{in } \mathcal{L}^\gamma(G_\varepsilon) \cap \mathcal{L}^\beta(G_\varepsilon) \cap \mathcal{L}^1(G_\varepsilon),$$
$$\boldsymbol{m}_0^R/\sqrt{\rho_0^R} \to \boldsymbol{m}_0/\sqrt{\rho_0} \quad \text{in } \mathcal{L}^2(G_\varepsilon). \tag{4.16}$$

Denote by $(\rho^R, \boldsymbol{u}^R)$ the solution of (2.1)–(2.3), (4.10) with the initial data $(\rho_0^R, \boldsymbol{u}_0^R)$ obtained in Lemma 4.2. We extend $(\rho^R, \boldsymbol{u}^R)$ to the domain $(\varepsilon, \infty) \times \mathbb{R}$ as follows:

$$\tilde{\rho}^R(t,r,z) := \begin{cases} \rho^R(t,r,z), & (r,z) \in \overline{\Omega}_R, \\ 0, & \text{otherwise}, \end{cases}$$
$$\tilde{\boldsymbol{u}}^R(t,r,z) := \begin{cases} \boldsymbol{u}^R(t,r,z), & (r,z) \in \overline{\Omega}_R, \\ \boldsymbol{u}^R(t,r,R), & \varepsilon < r < R,\ \pm z \geqslant R, \\ \boldsymbol{u}^R(t,R,z), & r \geqslant R,\ |z| < R, \\ 0, & \text{otherwise}. \end{cases}$$

Then, from (4.11) and (4.16) as well as (5.91) in Lions' book [12, p. 43], it follows that $\tilde{\boldsymbol{u}}^R \in L^2((0,T), \mathcal{H}^1_{\mathrm{loc}}(\overline{G}_\varepsilon))$ and

$$\|\tilde{\rho}^R\|_{L^\infty((0,T),\mathcal{L}^\gamma(G_\varepsilon)\cap\mathcal{L}^\beta(G_\varepsilon))} + \left\|\sqrt{\tilde{\rho}^R}\,\tilde{\boldsymbol{u}}^R\right\|_{L^\infty((0,T),\mathcal{L}^2_{\mathrm{loc}}(\overline{G}_\varepsilon))} + \|\tilde{\boldsymbol{u}}^R\|_{L^2((0,T),\mathcal{H}^1_{\mathrm{loc}}(\overline{G}_\varepsilon))} \leqslant C$$

with C being independent of R. Hence, as $R \to \infty$,

$$\tilde{\rho}^R \rightharpoonup \rho \quad \text{weak-}* \text{ in } L^\infty\big((0,T), \mathcal{L}^\gamma(G_\varepsilon) \cap \mathcal{L}^\beta(G_\varepsilon)\big),$$
$$\tilde{\boldsymbol{u}}^R \rightharpoonup \boldsymbol{u} \quad \text{weakly in } L^2\big((0,T), \mathcal{H}^1_{\mathrm{loc}}(\overline{G}_\varepsilon)\big).$$

On the other hand, utilizing (4.11), we argue exactly in the same way as in the derivation of (2.14) (with $\varphi(r) \equiv 1$) to deduce that there is a constant $C > 0$ independent of R, such that for all R large enough,

$$\int_0^T \int_K \left\{ \left(\rho^R\right)^{\gamma+\theta} + \varepsilon^\lambda \left(\rho^R\right)^{\beta+\theta} \right\} r\,\mathrm{d}r\,\mathrm{d}z\,\mathrm{d}t \leqslant C, \quad \theta = \gamma - 1, \tag{4.17}$$

for any compact set $K \subset \mathbb{R}^+ \times \mathbb{R}$.

On the other hand, taking into account that in any compact set of G_ε, $(\tilde{\rho}^R, \tilde{\boldsymbol{u}}^R) = (\rho^R, \boldsymbol{u}^R)$ for R sufficiently large, thus by the same proof as that of Lemma 2.3 in [5], we find that $(\tilde{\rho}^R, \tilde{\boldsymbol{u}}^R)$ satisfies (1.5) in $\mathcal{D}'((0,T) \times G_\varepsilon)$ with (ρ, u) replaced by $(\tilde{\rho}^R, \tilde{\boldsymbol{u}}^R)$. Thus, making use of (4.11) and (4.17), following the same procedure as in the proof of precompactness in Section 3, we find, by taking $R \to \infty$ in (2.1)–(2.3) and (4.10), that the weak limit $(\rho, \boldsymbol{u})$ is indeed a weak solution of (2.1)–(2.5). Moreover, by the lower semicontinuity of weak convergence, (4.11) and (4.16), we see that for any $l > 0$,

$$\sup_{t\in[0,T]}\int_{\Omega_l}\left(\frac{\rho|\boldsymbol{u}|^2}{2}+\frac{a\rho^\gamma}{\gamma-1}+\frac{\varepsilon^\lambda\rho^\beta}{\beta-1}\right)r\,\mathrm{d}r\,\mathrm{d}z+\mu\int_0^T\int_{\Omega_l}\left(|\nabla\boldsymbol{u}|^2+\frac{u_1^2}{r^2}\right)r\,\mathrm{d}r\,\mathrm{d}z\,\mathrm{d}t$$
$$\leqslant\liminf_{R\to\infty}E_R\left(\rho_0^R,\boldsymbol{m}_0^R\right)\leqslant\int_{G_\varepsilon}\left(\frac{|\boldsymbol{m}_0|^2}{2\rho_0}+\frac{a\rho_0^\gamma}{\gamma-1}+\frac{\varepsilon^\lambda\rho_0^\beta}{\beta-1}\right)r\,\mathrm{d}r\,\mathrm{d}z,$$

where $\Omega_l := (\varepsilon, l)\times(-l, l)$. Hence, (4.14) holds by Fatou's Lemma.

Finally, if we apply Lemma C.1 of [11, Appendix C] and Eq. (2.1), we find that $\rho^R \in C^0([0,T], L^\gamma_{\mathrm{loc}}(\Omega_R) - w)$. So, integrating (2.1) over $(0,t)\times\Omega_R$ and taking into account (4.10), we infer (cf. the proof of (5.50)–(5.51) in [12, p. 22])

$$\sup_{[0,T]}\int_{\Omega_R}\tilde{\rho}^R r\,\mathrm{d}r\,\mathrm{d}z=\sup_{[0,T]}\int_{\Omega_R}\rho^R r\,\mathrm{d}r\,\mathrm{d}z\leqslant\int_{\Omega_R}\rho_0^R r\,\mathrm{d}r\,\mathrm{d}z\leqslant\int_{G_\varepsilon}\rho_0 r\,\mathrm{d}r\,\mathrm{d}z.$$

Consequently, one has with the help of (4.16) that for any $l>0$,

$$\sup_{[0,T]}\int_{\Omega_l}\rho(t,r,z)r\,\mathrm{d}r\,\mathrm{d}z\leqslant\liminf_{R\to\infty}\int_{\Omega_l}\rho^R r\,\mathrm{d}r\,\mathrm{d}z\leqslant\int_{G_\varepsilon}\rho_0 r\,\mathrm{d}r\,\mathrm{d}z,$$

which yields (4.15). This completes the proof. □

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